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Josef Šlapal

REFLECTIVE AND COREFLECTIVE MODIFICATIONS OF THE CONSTRUCT OF TOPOLOGICAL SPACES WITHOUT AXIOMS

Generalized topologies obtained by replacing the Kuratowski closure axioms by some weaker ones occur in various branches of mathematics (see e.g. [5]) and have been studied by many mathematicians (see [3]). The most general of them — the topologies without axioms — are dealt with in this contribution. The upper and lower modifications of topologies without axioms with regard to the axioms O, I, M, A, U, K, B^*, B, S are described in [3]. In [4] there are determined those axioms which are preserved by the individual modifications. By the help of [3] and [4], in the present note we solve the problem of determining the axioms that give reflective or coreflective modifications of the construct (i.e. concrete category of structured sets and structure-compatible maps) of topological spaces without axioms. This is done by finding the axioms with regard to which upper (lower) modifications are reflections (coreflections).

The categorical terminology used is taken from [1]. If $\mathcal{I}$ is a construct and $(X,\alpha) \in \mathcal{I}$ an object, then by saying that (X,β) is a reflection [coreflection] of (X,α) in a subconstruct $\mathcal{T}$ of $\mathcal{I}$ we mean that $\mathrm{id}_X : (X,\alpha) \to (X,\beta)[\mathrm{id}_X : (X,\beta) \to (X,\alpha)]$ is a reflection [coreflection] of (X,α) in $\mathcal{T}$. A reflective (coreflective) modification of a construct $\mathcal{I}$ is then any full subconstruct of $\mathcal{I}$ in which each object of $\mathcal{I}$ has a reflection (coreflection). Let $\mathcal{I}$ be a fibre-small construct, $\mathcal{T}$ a subconstruct of $\mathcal{I}$ and $(X,\alpha) \in \mathcal{I}$ an object. An object $(X,\beta) \in \mathcal{T}$ is called the upper [lower] modification of (X,α) in $\mathcal{T}$, if β is the finest [coarsest] of all $\mathcal{T}$-structures on X that are coarser [finer] than α. It is well known (see e.g. [2], Notes) that each reflection [coreflection] of (X,α) in $\mathcal{T}$ is the upper [lower] modification of (X,α) in $\mathcal{T}$. On the other hand, obviously, if (X,β) is the upper [lower] modification of (X,α) in $\mathcal{T}$, then (X,β) is a reflection [coreflection] of (X,α) in $\mathcal{T}$ iff for each object

$(Y,\gamma) \in \mathcal{T}$ and each morphism $f:(X,\alpha) \to (Y,\gamma)[f:(Y,\gamma) \to (X,\alpha)]$ in $\mathcal{I}$ the map $f:(X,\beta) \to (Y,\gamma)[f:(Y,\gamma) \to (X,\beta)]$ is a morphism in $\mathcal{T}$.

By a topology without axioms (briefly a topology) on a set X we understand a map $u: \exp X \to \exp X$. The pair (X,u) is then called a topological space without axioms (briefly a topological space). Given two topological spaces (X,u) and (Y,v), a map $f:(X,u) \to (Y,v)$ is called continuous if $f(uA) \subseteq vf(A)$ for each subset $A \subseteq X$. The construct of all topological spaces and continuous maps will be denoted by $\mathcal{P}$. Obviously, $\mathcal{P}$ is fibre-small (and even fibbre-complete). From the literature listed in [3] we take the following nine axioms for topologies u on a given set X:

1. $u\emptyset = \emptyset$ — O – axiom,
2. $A \subseteq X \Rightarrow A \subseteq uA$ — I – axiom,
3. $A \subseteq B \subseteq X \Rightarrow uA \subseteq uB$ — M – axiom,
4. $A, B \subseteq X \Rightarrow u(A \cup B) \subseteq uA \cup uB$ — A – axiom,
5. $A \subseteq X \Rightarrow uuA \subseteq uA$ — U – axiom,
6. $x, y \in X, x \in u\{y\}, y \in u\{x\} \Rightarrow x = y$ — K – axiom,
7. $x, y \in X, x \in u\{y\} \Rightarrow y \in u\{x\}$ — B* – axiom,
8. $x \in X \Rightarrow u\{x\} \subseteq \{x\}$ — B – axiom,
9. $\emptyset \neq A \subseteq X \Rightarrow uA \subseteq \bigcup_{x \in A} u\{x\}$ — S – axiom.

Many topologies studied by individual authors are exactly the topologies fulfilling some of these nine axioms — see [3]. Let $\lambda, \mu \in \{O, I, M, A, U, K, B^*, B, S\}$. A topology u on a set X and the topological space (X,u) are called λ-topology and λ-space if u fulfils the λ-axiom; they are called $\lambda\mu$-topology and $\lambda\mu$-space if u fulfils both the λ-axiom and the μ-axiom (we speak about $\lambda\mu$-axiom); etc. By $\mathcal{P}_\lambda$ we denote the full subconstruct of $\mathcal{P}$ whose objects are precisely the λ-spaces, by $\mathcal{P}_{\lambda\mu}$ the full subconstruct of $\mathcal{P}$ whose objects are precisely the $\lambda\mu$-spaces, etc. If $(X,u) \in \mathcal{P}$ and (X,v) is the upper [lower] modification of (X,u) in $\mathcal{P}_\lambda$, then v is called the upper [lower] λ-modification of u and denoted by $u^\lambda[u_\lambda]$. Analogically the term of the lower [upper] $\lambda\mu$-modification of a topology can be introduced.

THEOREM 1. a) *$\mathcal{P}_0$ is the only subconstruct of $\mathcal{P}$ whose reflective modification is $\mathcal{P}_0$, and $\mathcal{P}_{0\lambda}$ is the only subconstruct of $\mathcal{P}_\lambda$ whose reflective modification is $P_{0\lambda}$ whenever $\lambda \in \{I, M, A, U, K, B^*, B, S\}$.*

b) *$\mathcal{P}_0$ is a coreflective modification of $\mathcal{P}$, and $\mathcal{P}_{0\lambda}$ is a coreflective modification of $\mathcal{P}_\lambda$ whenever $\lambda \in \{I, M, A, U, K, B^*, B, S\}$.*

Proof. a) Let $(X,u) \in \mathcal{P}$ be an object and let u be not an O-topology. Then there does not exist any O-topology on X coarser than u. Conse-

quently, there exists no reflection of (X,u) in $\mathcal{P}_0$, and if $(X,u) \in P_\lambda$, then there exists no reflection of (X,u) in $\mathcal{P}_{0\lambda}$ whenever $\lambda \in \{I, M, A, U, K, B^*, B, S\}$.

b) Let $(X,u) \in \mathcal{P}$ be an object and let v be the topology on X given by

$$v\emptyset = \emptyset,$$
$$\emptyset \neq A \subseteq X \Rightarrow vA = uA.$$

Obviously, $v = u_0$. Now, if $(Y,w) \in \mathcal{P}_0$ is an object and if $f : (Y,w) \to (X,u)$ is a continuous map, then clearly $f : (Y,w) \to (X,v)$ is also continuous. Thus (X,v) is a coreflection of (X,u) in $\mathcal{P}_0$. In [4] it is shown that the lower O-modification preserves each of the axioms I, M, A, U, K, B^*, B, S.

THEOREM 2. a) *$\mathcal{P}_I$ is a reflective modification of $\mathcal{P}$, and $\mathcal{P}_{I\lambda}$ is a reflective modification of $\mathcal{P}_\lambda$ whenever $\lambda \in \{O, M, A, AU, K, B^*, B, S\}$.*

b) *$\mathcal{P}_I$ is the only subconstruct of $\mathcal{P}$ whose coreflective modification is $\mathcal{P}_I$, and $\mathcal{P}_{I\lambda}$ is the only subconstruct of $\mathcal{P}_\lambda$ whose coreflective modification is $\mathcal{P}_{I\lambda}$ whenever $\lambda \in \{O, M, A, U, K, B^*, B, S\}$.*

Proof. a) Let $(X,u) \in \mathcal{P}$ be an object and let v be the topology on X given by

$$A \subseteq X \Rightarrow vA = uA \cup A.$$

Clearly, $v = u^I$. Let $(Y,w) \in \mathcal{P}_I$ be an object and let $f : (X,u) \to (Y,w)$ be a continuous map. Then $f : (X,v) \to (Y,w)$ is continuous because $f(vA) = f(uA) \cup f(A) \subseteq wf(A)$ holds for any subset $A \subseteq X$. Thus (X,v) is a reflection of (X,u) in $\mathcal{P}_I$. In [4] it is shown that the upper I-modification preserves each of the axioms $O, M, A, AU, K, B^*, B, S$.

b) Let $(X,u) \in \mathcal{P}$ be an object and let u be not an I-topology. Then there does not exist any I-topology on X finer than u. Consequently, there exists no coreflection of (X,u) in $\mathcal{P}_I$, and if $(X,u) \in \mathcal{P}_\lambda$, then there exists no coreflection of (X,u) in $\mathcal{P}_{I\lambda}$ whenever $\lambda \in \{O, M, A, U, K, B^*, B, S\}$.

THEOREM 3. a) *$\mathcal{P}_M$ is a reflective modification of $\mathcal{P}$, and $\mathcal{P}_{M\lambda}$ is a reflective modification of $\mathcal{P}_\lambda$ whenever $\lambda \in \{O, I, A, OK, OB^*, IB^*, OB, IB, S\}$.*

b) *$\mathcal{P}_M$ is not a coreflective modification of $\mathcal{P}$.*

Proof. a) Let $(X,u) \in \mathcal{P}$ be an object and let v be the topology on X given by

$$A \subseteq X \Rightarrow vA = \bigcup_{B \subseteq A} uB.$$

By [3], $v = u^M$. Let $(Y,w) \in \mathcal{P}_M$ and let $f : (X,u) \to (Y,w)$ be a continuous map. Then for any subset $A \subseteq X$ we have $f(vA) = f(\bigcup_{B\subseteq A} uB) = \bigcup_{B\subseteq A} f(uB) \subseteq \bigcup_{B\subseteq A} wf(B) = wf(A)$. Thus $f : (X,v) \to (Y,w)$ is continuous. Therefore (X,v) is a reflection of (X,u) in $\mathcal{P}_M$. In [4] it is shown that the

upper M-modification preserves each of the axioms O, I, A, OK, IK, OB^*, IB^*, OB, IB, S.

b) Let $(X, u) \in \mathcal{P}$ be an object and let v be the topology on X given by

$$A \subseteq X \Rightarrow vA = \bigcap_{A \subseteq B \subseteq X} uB.$$

By [3], $v = u_M$. Put $X = \{x, y\}$, $u\emptyset = \emptyset = uX$, $u\{x\} = \{x\}$, $u\{y\} = \{y\}$. Let Y be a singleton and put $w\emptyset = \emptyset, wY = Y$. Then $(Y, w) \in \mathcal{P}_M$. Let $f : Y \to X$ be the map with $f(Y) = \{x\}$. Then $f : (Y, w) \to (X, u)$ is clearly continuous. But $f : (Y, w) \to (X, v)$ is not continuous because $f(wY) = \{x\} \not\subseteq \emptyset = vf(Y)$. Hence (X, v) is not a coreflection of (X, u) in $\mathcal{P}_M$, i.e. (X, u) has no coreflection in $\mathcal{P}_M$.

Remark 1. Of course, there exist subbconstructs of $\mathcal{P}$ different from $\mathcal{P}_M$ whose coreflective modifications are $\mathcal{P}_M$. An example of such a subconstruct is the full subconstruct of $\mathcal{P}$ whose objects are precisely the objects of $\mathcal{P}_M$ and the topological spaces (X, u) with card $X \geq 1$, $u\emptyset = X$ and $\emptyset \neq A \subseteq X \Rightarrow uA = \emptyset$.

THEOREM 4. a) *$\mathcal{P}_A$ is the only subconstruct of $\mathcal{P}$ whose reflective modification is $\mathcal{P}_A$, and $\mathcal{P}_{A\lambda}$ is the only subconstruct of $\mathcal{P}_\lambda$ whose reflective modification is $\mathcal{P}_{A\lambda}$ whenever $\lambda \in \{I, S\}$.*

b) *$\mathcal{P}_A$ is a coreflective modification of $\mathcal{P}$, and $\mathcal{P}_{A\lambda}$ is a coreflective modification of $\mathcal{P}_A$ whenever $\lambda \in \{O, I, M, MU, K, B^*, B, S\}$.*

Proof. a) In [3] it is proved that if $(X, u) \in \mathcal{P}$, then the upper A-modification of u exists iff u is an A-topology. But it can be proved similarly that if $\lambda \in \{I, S\}$ and $(X, u) \in \mathcal{P}_\lambda$, then the upper $A\lambda$-modification exists iff u is an A-topology. Consequently, if $(X, u) \in \mathcal{P}$ is an object, then an A-space (X, v) is a reflection of (X, u) in $\mathcal{P}_A$ iff $u = v$, and if $\lambda \in \{I, S\}$ and $(X, u) \in \mathcal{P}_\lambda$ is an object, than an $A\lambda$-space (X, v) is a reflection of (X, u) in $\mathcal{P}_{A\lambda}$ iff $u = v$.

b) Let $(X, u) \in \mathcal{P}$ be an object and let v be the topology on X given by

$$A \subseteq X \Rightarrow vA \bigcap \Big\{B \subseteq X | B = \bigcup_{i=1}^{m} uA_i, \bigcup_{i=1}^{m} A_i = A, m \text{ is a positive integer}\Big\}.$$

By [3], $v = u_A$. Let $(Y, w) \in \mathcal{P}_A$ be an object and let $f : (Y, w) \to (X, u)$ be a continuous map. Let $A \subseteq Y$ be a subset, let m be a positive integer and let $\{A_i | i = 1, \ldots, m\}$ be a system of subsets of X with $\bigcup_{i=1}^m A_i = f(A)$. Then $A = \bigcup_{i=1}^m (f^{-1}(A_i) \cap A)$ and thus $wA \subseteq \bigcup_{i=1}^m w(f^{-1}(A_i) \cap A)$. Hence $f(wA) \subseteq \bigcup_{i=1}^m f(w(f^{-1}(A_i) \cap A)) \subseteq \bigcup_{i=1}^m uf(f^{-1}(A_i) \cap A) = \bigcup_{i=1}^m uA_i$. Consequently, $f(wA) \subseteq \bigcap \{B \subseteq X | B = \bigcup_{i=1}^m uA_i, \bigcup_{i=1}^m A_i = f(A), m$ is a positive integer$\} = vf(A)$. Therefore $f : (Y, w) \to (X, v)$ is a continuous

map. Thus (X,v) is a coreflection of (X,u) in $\mathcal{P}_A$. In [4] it is shown that the lower A-modification preserves each of the axioms $O, I, M, MU, K, B^*, B, S$.

THEOREM 5. a) *$\mathcal{P}_{MU}$ is a reflective modification of $\mathcal{P}_M$, and $\mathcal{P}_{MU\lambda}$ is a reflective modification of $\mathcal{P}_{M\lambda}$ whenever $\lambda \in \{O, I, A, K, B^*, B, S\}$.*

b) *$\mathcal{P}_{MU}$ is the only subconstruct of $\mathcal{P}_M$ whose coreflective modification is $\mathcal{P}_{MU}$, and $\mathcal{P}_{MU\lambda}$ is the only subconstruct of $\mathcal{P}_{M\lambda}$ whose coreflective modification is $\mathcal{P}_{MU\lambda}$ whenever $\lambda \in \{O, B\}$.*

Proof. a) Let $(X,u) \in \mathcal{P}$ be an M-space and let v be the topology on X given by

$$A \subseteq X \Rightarrow vA = \bigcap\{B \subseteq X | uA \subseteq B, uB \subseteq B\}.$$

By [3], $v = u^U$. Let $(Y,w) \in \mathcal{P}_U$ be an object and let $f : (X,u) \to (Y,w)$ be a continuous map. Let $A \subseteq X$ be a subset. There holds $uA \subseteq f^{-1}(f(uA)) \subseteq f^{-1}(wf(A))$. Next, we have $f(uf^{-1}(wf(A))) \subseteq wf(f^{-1}(wf(A))) = wwf(A) \subseteq wf(A)$, hence $uf^{-1}(wf(A)) \subseteq f^{-1}(f(uf^{-1}(wf(A)))) \subseteq f^{-1}(wf(A))$. Consequently, $vA \subseteq f^{-1}(wf(A))$. Thus $f(vA) \subseteq f(f^{-1}(wf(A))) = wf(A)$, so that $f : (X,v) \to (Y,w)$ is continuous. Hence (X,v) is a reflection of (X,u) in $\mathcal{P}_U$. In [4] it is shown that u^U is an M-topology and that the upper U-modification of M-tolopogies preserves each of the axioms O, I, A, K, B^*, B, S.

b) In [3] it is proved that if $(X,u) \in \mathcal{P}_M$ is an object, then the lower U-modification of u exists iff u is a U-topology. From this it follows that if $\lambda \in \{O, B\}$ and $(X,u) \in \mathcal{P}_{M\lambda}$, then the lower $U\lambda$-modification of u exists iff u is a U-topology (because v is a λ-topology whenever v is a topology on X finer than u). Consequently, if $(X,u) \in \mathcal{P}_M$, then an MU-space (X,v) is a coreflection of (X,u) in $\mathcal{P}_{MU}$ iff $u = v$, and if $\lambda \in \{O, B\}$ and $(X,u) \in \mathcal{P}_{M\lambda}$, then an $MU\lambda$-space (X,v) is a coreflection of (X,u) in $\mathcal{P}_{MU\lambda}$ iff $u = v$.

Remark 2. We shall show that $\mathcal{P}_U$ is neither reflective nor coreflective modification of $\mathcal{P}$. On that account, let $X = \{x, y\}$ and put $u_1\{x\} = \{x\}, u_2\{x\} = \{y\}$ and $u_1 A = u_2 A = \{y\}$ whenever $\{x\} \neq A \subseteq X$. Then $(X, u_1), (X, u_2) \in \mathcal{P}_U$ and u_1, u_2 are incomparable, i.e. u_1 is neither finer nor coarser than u_1. Next, put $v\{x\} = \emptyset, w\{x\} = \{x, y\}$ and $vA = wA = \{y\}$ whenever $\{x\} \neq A \subseteq X$. Then clearly v is finer and w is coarser than both u_1 and u_2. Thus there exists no upper U-modification of v because there is no U-topology on X that is coarser than v and finer than both u_1 and u_2. Similarly, there exists no lower U-modification of w because there is no U-topology on X that is finer than w and coarser than both u_1 and u_2. Consequently, there does not exist any reflection of (X,v) in $\mathcal{P}_U$, and there does not exist any coreflection of (X,w) in $\mathcal{P}_U$.

THEOREM 6. a) *$\mathcal{P}_K$ is the only subconstruct of $\mathcal{P}$ whose reflective modification is $\mathcal{P}_K$, and $\mathcal{P}_{K\lambda}$ is the only subconstruct of $\mathcal{P}_\lambda$ whose reflective modification is $\mathcal{P}_{K\lambda}$ whenever $\lambda \in \{O, I, M, A, U, B^*, B, S\}$.*

b) *$\mathcal{P}_K$ is the only subconstruct of $\mathcal{P}$ whose coreflective modification is $\mathcal{P}_K$, and $\mathcal{P}_{K\lambda}$ is the only subconstruct of $\mathcal{P}_\lambda$ whose coreflective modification is $\mathcal{P}_{K\lambda}$ whenever $\lambda \in \{O, B\}$.*

Proof. a) Let $(X, u) \in \mathcal{P}$ be an object and let u be not a K-topology. Then there does not exist any K-topology on X coarser than u. Consequently, there exists no reflection of (X, u) in $\mathcal{P}_K$, and if $(X, u) \in \mathcal{P}_\lambda$, then there exists no reflection of (X, u) in $\mathcal{P}_{K\lambda}$ whenever $\lambda \in \{O, I, M, A, U, B^*, B, S\}$.

b) In [3] it is proved that if $(X, u) \in \mathcal{P}$, then the lower K-modification of u exists iff u is a K-topology. But this implies that if $\lambda \in \{O, B\}$ and $(X, u) \in \mathcal{P}_\lambda$, then the lower $K\lambda$-modification of u exists iff u is a K-topology. Consequently, if $(X, u) \in \mathcal{P}_M$, then a K-space (X, v) is a coreflection of (X, u) in $\mathcal{P}_K$ iff $u = v$, and if $\lambda \in \{O, B\}$ and $(X, u) \in \mathcal{P}_\lambda$, then a $K\lambda$-space (X, v) is a coreflection of (X, u) in $\mathcal{P}_{K\lambda}$ iff $u = v$.

THEOREM 7. a) *$\mathcal{P}_{B^*}$ is a reflective modification of $\mathcal{P}$, and $\mathcal{P}_{B^*\lambda}$ is a reflective modification of $\mathcal{P}_\lambda$ whenever $\lambda \in \{O, I, M, K, B\}$.*

b) *$\mathcal{P}_{B^*}$ is a coreflective modification of $\mathcal{P}$, and $\mathcal{P}_{B^*\lambda}$ is a coreflective modification of $\mathcal{P}_\lambda$ whenever $\lambda \in \{O, I, A, B, S\}$.*

Proof. a) Let $(X, u) \in P$ be an object and let v be the topology on X given by

$$x \in X \Rightarrow v\{x\} = u\{x\} \cup \{y \in X | x \in u\{y\}\},$$
$$A \subseteq X, \text{ card } A \neq 1 \Rightarrow vA = uA.$$

By [3], $v = u^{B^*}$. Let $(Y, w) \in \mathcal{P}_{B^*}$ be an object and let $f : (X, u) \to (Y, w)$ be a continuous map. Let $x \in X$ be a point. Then $f(v\{x\}) = f(u\{x\}) \cup \{f(y) | y \in X, x \in u\{y\}\} \subseteq w\{f(x)\} \cup \{f(y) | y \in X, f(x) \in f(u\{y\})\} \subseteq w\{f(x)\} \cup \{f(y) | y \in X, f(x) \in w\{f(y)\}\} = w\{f(x)\} \cup \{f(y) | y \in X, f(y) \in w\{f(x)\}\} \subseteq w\{f(x)\} \cup \{z \in Y | z \in w\{f(x)\}\} = wf(\{x\})$. Therefore $f : (X, v) \to (Y, w)$ is continuous, and thus (X, v) is a reflection of (X, u) in $\mathcal{P}_{B^*}$. In [4] it is shown that the upper B^*-modification preserves each of the axioms O, I, M, K, B.

b) Let $(X, u) \in \mathcal{P}_{B^*}$ be an object and let v be the topology on X given by

$$x \in X \Rightarrow v\{x\} = u\{x\} \cap \{y \in X | x \in u\{y\}\},$$
$$A \subseteq X, \text{ card } A \neq 1 \Rightarrow vA = uA.$$

By [3], $v = u_{B^*}$. Let $(Y, w) \in \mathcal{P}_{B^*}$ be an object and let $f : (Y, w) \to (X, u)$ be a continuous map. Let $x \in Y$ be a point. Then $f(w\{x\}) = \{y \in X | y \in f(w\{x\})\} = \{y \in X | \exists z \in w\{x\} : y = f(z)\} = \{y \in X | \exists z \in Y : x \in w\{z\},\, y = f(z)\} \subseteq \{y \in X | \exists z \in Y : f(x) \in f(w\{z\}),\, y = f(z)\} \subseteq \{y \in X | \exists z \in Y : f(x) \in u\{f(z)\},\, y = f(z)\} \subseteq \{y \in X | f(x) \in u\{y\}\}$. Next, since $f : (Y, w) \to (X, u)$ is continuous, we have $f(w\{x\}) \subseteq u\{f(x)\}$. Hence $f(w\{x\}) \subseteq u\{f(x)\} \cap \{y \in X | f(x) \in u\{y\}\} = v\{f(x)\}$. Therefore $f : (Y, w) \to (X, v)$ is continuous, and thus (X, v) is a coreflection of (X, u) in $\mathcal{P}_{B^*}$. In [4] it is shown that the lower B^*-modification preserves each of the axioms O, I, A, B, S.

THEOREM 8. a) *$\mathcal{P}_B$ is the only subconstruct of $\mathcal{P}$ whose reflective modification is $\mathcal{P}_B$, and $\mathcal{P}_{B\lambda}$ is the only subconstruct of $\mathcal{P}_\lambda$ whose reflective modification is $\mathcal{P}_{B\lambda}$ whenever $\lambda \in \{O, I, M, A, U, K, B^*, S\}$.*

b) *$\mathcal{P}_B$ is a coreflective modification of $\mathcal{P}$, and $\mathcal{P}_{B\lambda}$ is a coreflective modification of $\mathcal{P}_\lambda$ whenever $\lambda \in \{O, I, OM, OU, K, B^*\}$.*

Proof. a) Let $(X, u) \in \mathcal{P}$ be an object and let u be not a B-topology. Then there does not exist any B-topology on X coarser than u. Consequently, there exists no reflection of (X, u) in $\mathcal{P}_B$, and if $(X, u) \in \mathcal{P}_\lambda$, then there exists no reflection of (X, u) in $\mathcal{P}_{B\lambda}$ whenever $\lambda \in \{O, I, M, A, U, K, B^*, S\}$.

b) Let $(X, u) \in \mathcal{P}$ be an object and let v be the topology on X given by

$$x \in X \Rightarrow v\{x\} = \{x\} \cap u\{x\},$$
$$A \subseteq X,\ \operatorname{card} A \neq 1 \Rightarrow vA = uA.$$

By [3], $v = u_B$. Let $(Y, w) \in \mathcal{P}_B$ and let $f : (Y, w) \to (X, u)$ be a continuous map. Let $x \in Y$ be a point. Then $f(w\{x\}) \subseteq f(\{x\})$ and $f(w\{x\}) \subseteq uf(\{x\})$. Hence $f(w\{x\}) \subseteq \{f(x)\} \cap u\{f(x)\} = v\{f(x)\}$. Consequently, $f : (Y, w) \to (X, u)$ is continuous. Thus (X, v) is a coreflection of (X, u) in $\mathcal{P}_B$. In [4] it is shown that the lower B-modification preserves each of the axioms O, I, OM, OU, K, B^*.

THEOREM 9. a) *$\mathcal{P}_{MS}$ is the only subconstruct of $\mathcal{P}_M$ whose reflective modification is $\mathcal{P}_{MS}$, and $\mathcal{P}_{MS\lambda}$ is the only subconstruct of $\mathcal{P}_{M\lambda}$ whose reflective modification is $\mathcal{P}_{MS\lambda}$ whenever $\lambda \in \{I, A\}$.*

b) *$\mathcal{P}_S$ is a coreflective modification of $\mathcal{P}$, and $\mathcal{P}_{S\lambda}$ is a coreflective modification of $\mathcal{P}_\lambda$ whenever $\lambda \in \{O, I, M, A, MU, K, B^*, B\}$.*

Proof. a) In [3] it is proved that if $(X, u) \in \mathcal{P}_M$, then the upper S-modification of u exists iff u is an S-topology. But it can be proved similarly that if $\lambda \in \{I, A\}$ and $(X, u) \in \mathcal{P}_{M\lambda}$, then the upper $S\lambda$-modification of u exists iff u is an S-topology. Consequently, if $(X, u) \in \mathcal{P}_M$, then an MS-

space (X,v) is a reflection of (X,u) in $\mathcal{P}_{MS}$ iff $u=v$, and if $\lambda \in \{I,A\}$ and $(X,u) \in \mathcal{P}_{M\lambda}$, then an $MS\lambda$-space (X,v) is a reflection of (X,u) in $\mathcal{P}_{MS\lambda}$ iff $u=v$.

b) Let $(X,u) \in \mathcal{P}$ be an object and let v be the topology on X given by

$$v\emptyset = u\emptyset,$$
$$\emptyset \neq A \subseteq X \Rightarrow vA = \bigcup_{x\in A} u\{x\} \cap uA.$$

It can be easily proved that $v = u_S$ (in [3] this equality is proved only for the case when u is an M-topology). Let $(Y,w) \in \mathcal{P}_S$ be an object and let $f:(Y,w) \to (X,u)$ be a continuous map. Let $\emptyset \neq A \subseteq X$ be a subset. Then $f(wA) \subseteq uf(A)$ and $f(wA) \subseteq f(\bigcup_{x\in A} w\{x\}) = \bigcup_{x\in A} f(w\{x\}) \subseteq \bigcup_{x\in A} u\{f(x)\} = \bigcup_{y\in f(A)} u\{y\}$. Consequently, $f(wA) \subseteq \bigcup_{y\in f(A)} u\{y\} \cap uf(A) = vf(A)$. Therefore $f:(Y,w) \to (X,v)$ is continuous, and thus (X,v) is a coreflection of (X,u) in $\mathcal{P}_S$. It is easy to show that the lower S-modification preserves each of the axioms $O, I, M, A, MU, K, B^*, B$.

Remark 3. We shall show that $\mathcal{P}_S$ is not a reflective modification of $\mathcal{P}$. On that account, let $X=\{x,y\}$ and put $u_1\emptyset = u_2\emptyset = \emptyset$, $u_1\{x\} = u_2\{y\} = X$, $u_1\{y\} = u_2\{x\} = \{y\}$, $u_1X = u_2X = X$. Then $(X,u_1),(X,u_2) \in \mathcal{P}_S$ and u_1, u_2 are incomparable. Put $v\emptyset = v\{x\} = v\{y\} = \emptyset, vX = X$. Then v is finer than both u_1 and u_2. Thus there exists no upper S-modification of v because there is no S-topology on X that is coarser than v and finer than both u_1 and u_2. Consequently, there does not exist any reflection of (X,v) in $\mathcal{P}_S$.

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ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF A DELAYED DIFFERENTIAL EQUATION

1. Introduction

In this paper we consider the asymptotic expansion of solutions of delayed differential equations

$$g(t)\dot{y}(t) = -ay(t) + \sum_{i+j=2}^{N} c_{ij}(t)y^i(t)y^j(t-r), \tag{1}$$

where $N \geq 2$ is an integer, $a > 0,\ r > 0$ are constants, $g(t) : R_+^0 \to R_+$, $c_{ij}(t) : R_+^0 \to R$ are continuous functions (further conditions will be given latter). The purpose of this paper is to prove that for each real parameter C and function $\psi \in B_o = \{\psi \in C^0[-r,0], \|\psi\| \leq 1,\ \psi(0) = 0\}$ which describe the power of the set of solutions, there is a solution $y(t) = y(t, C, \psi)$ of 1 which may be at $t \to \infty$ represented by asymptotic series (symbol $\approx$ denotes the asymptotic expansions)

$$y(t, C, \psi) \approx \sum_{k=1}^{\infty} f_k(t)\varphi^k(t, C) \tag{2}$$

where $\varphi(t, C)$ is the solution of equation $g(t)\dot{y}(t) = -ay(t)$, given by the formula $\varphi(t, C) = C \exp \int_0^t \frac{-a}{g(u)}\, du$, $f_1(t) \equiv 1$ and the functions $f_k(t)$ for $k = 2, \ldots, n$ are particular solutions of some system of auxiliary differential equations. To prove our results we will use Ważewski's topological method in the form, proposed by K. Rybakowski [5], which may be used for differential equations with retarded arguments. The first Lyapunoff's method is often used to construct the solutions of ordinary differential equations in the form of power-like series. Such a way is not possible here. First lefthand ends of existence intervals of partial sums can tend to infinity and, secondly, if it does not happen, the partial sums need not to converge uniformly. The modification of the first Lyapunoff's method were used in [6], [1].

Delayed differential equations appear in many technical problems. The form of equation (1) include some equations which have been recently considered. For example the logistic equation with recruitment delays

$$\dot{x}(t) = x(t-r)(A - Bx(t))$$

which were considered by Gopalsamy [2], with regard to the applications on ecology. After substitution $x(t) = \frac{A}{B} + y(t)$ have the form of equation (1), where $g(t) = 1$, $a = A$, $c_{11} = -B$ and $c_{ij} = 0$ for $i \neq 1$, $j \neq 1$, $N = 2$. Moreover also one branch of the equation partially solved with respect to derivative in Diblik's work [1] have (after solving with respect to derivatives) the form of the equation (1), in which are not terms with retarded arguments.

2. Preliminaries

To describe simply coefficients of power series raised to a power, it is suitable to denote: α, β — are sequences of nonnegative integers with finite sumation.

Let $\alpha = \{\alpha_k\}_{k=1}^{\infty}$, then we denote

$$|\alpha| = \sum_{k=1}^{\infty} \alpha_k,\ V(\alpha) = \sum_{k=1}^{\infty} k\alpha_k,\ \alpha! = \prod_{k=1}^{\infty} \alpha_k!,\ \max(\alpha) = \max\{k \mid \alpha_k \neq 0\}.$$

Let $\mathbf{a} = \{a_k\}_{k=1}^{\infty}$ be any sequence (of numbers or functions). We define

$$\mathbf{a}^{\alpha} = \prod_{k=1}^{\infty} a_k^{\alpha_k}, \quad \text{where } a_k^0 = 1 \text{ for every } a_k.$$

Then it is possible to prove

$$\Big(\sum_{k=1}^{\infty} a_k \mathbf{x}^k\Big)^n = \sum_{k=n}^{\infty} \mathbf{x}^k \sum_{n}^{k} \frac{n!}{\alpha!}\mathbf{a}^{\alpha},$$

where $\sum_n^k$ denotes the sumation over all sequences such that $|\alpha| = n, V(\alpha) = k$. As we work with the product of the power series raised to a power, we denote $\sum_{i,j}^k$ is the sumation over all couples (α, β) such that $V(\alpha) + V(\beta) = k$, $|\alpha| = i$, $|\beta| = j$.

Throughout this paper $g(t)$, $G(t)$ denote functions such that

C1. $g(t) \in C^0[0,\infty)$, $g(t) > 0$ for $t \geq t_0$ and $g(t) = O(1)$ as $t \to \infty$.

C2. $G(t) = o(g(t))$ as $t \to \infty$, where $G(t) = (\int_0^t g^{-1}(u)\,du)^{-1}$

C3. there is a constant $\lambda > 0$ such that

$$\frac{g(t) - g(t-r)}{g(t-r)} = o(G^{\lambda}(t)) \quad \text{as } t \to \infty.$$

This condition enables us to consider relative large class of functions: $g(t)$ may be constant, a periodical function (r is a period) or there is a positive $\lim_{t\to\infty} g(t)$ and if $\lim_{t\to\infty} g(t) = 0$ in addition then the function $g(t)$ must satisfy

$$\int_0^t g(u)\,du = o(g^k(t)) \quad \text{as } t \to \infty,\ k > 0 \text{ is a constant.}$$

LEMMA 1. *Let functions $g(t)$, $G(t)$ satisfy the conditions* **C1, C2, C3**. *Then*:

1. $G(t) \sim G(t-K)$ *as* $t \to \infty$ *where* K *is any constant*
2. $g(t)(g^{-1}(t-ir) - g^{-1}(t-ir+r)) = o(G^\lambda(t))$ *as* $t \to \infty$.

Proof.

$$\lim_{t\to\infty} \frac{G(t)}{G(t-K)} = \lim_{t\to\infty} \frac{\int_0^t g^{-1}(u)\,du - \int_{t-K}^t g^{-1}(u)\,du}{\int_0^t g^{-1}(u)\,du} =$$

$$= 1 - \lim_{t\to\infty} G(t) \int_{t-K}^{t} g^{-1}(u)\,du = 1,$$

therefore the function $G(t)$ is a decraesing function and we obtain

$$\lim_{t\to\infty} G(t) \int_{t-K}^{t} g^{-1}(u)\,du \le \lim_{t\to\infty} \int_{t-K}^{t} G(u) g^{-1}(u)\,du \le K o(1) = 0.$$

Moreover for $t \to \infty$ using **C3** we get

$$g(t) = g(t-ir) \prod_{j=1}^{i} (1 + o(G^\lambda(t-jr))) = g(t-ir) \prod_{j=1}^{i} (1 + o(G^\lambda(t)(1 + o(1)))$$

$$= g(t-ir) \sum_{j=1}^{i} \binom{i}{j} 1^j (o(G^\lambda(t)(1+o(1))))^{i-j} = g(t-ir)(1 + o(G^\lambda(t)).$$

Thus

$$\frac{g(t) - g(t-ir)}{g(t-ir)} = o(G^\lambda(t)).$$

Eventually we get

$$g(t)(g^{-1}(t-ir) - g^{-1}(t-ir+r)) =$$

$$\frac{g(t) - g(t-ir)}{g(t-ir)} - \frac{g(t) - g(t-ir+r)}{g(t-ir+r)} = o(G^\lambda(t)).$$

LEMMA 2. *Let the coefficients of equation*

(3) $$g(t)\dot{y}(t) = Ky(t) + E(t)f(t)$$

satisfy:

1. *$K > 0$ is a constant,*
2. *the functions $g(t)$, $G(t) = (\int_0^t g^{-1}(u)\,du)^{-1}$ fulfill* **C1, C2, C3,**
3. *$E(t) \equiv \exp \sum_{i=1}^{n} \int_{t-ir}^{t-ir+r} \frac{K_i}{g(u)}\,du$, where $K_i > 0$ are constants,*
4. *the function $f(t)$ has the asymptotic form $f(t) = G^{\gamma}(t)b(t) + O(G^{\gamma+\varepsilon_1}(t))$, as $t \to \infty$ where $\varepsilon_1 > 0$, γ are constants, $b(t) \in C^1[t_0,\infty)$ and moreover: $b(t) = o(G^{\tau}(t))$ as $t \to \infty$, for all $\tau > 0$, $g(t)\dot{b}(t) = o(G^{\delta}(t))$ as $t \to \infty$, where $\delta > 0$ is a constant.*

Then there exists the solution $Y(t)$ of (3) such that, the following asymptotic relations hold

$$Y(t) = E(t)G^{\gamma}(t)\Big(-\frac{b(t)}{K} + O(G^{\varepsilon}(t))\Big) \quad \dot{Y}(t) = O(g^{-1}(t)G^{\gamma+\varepsilon}(t)),$$

where $0 < \varepsilon < \min(\lambda, \varepsilon_1, \delta, 1)$ is a constant.

Proof. After the subtitution $y(t) = x(t)E(t)$ the equation (3) has form:

$$(4)\ \ g(t)\dot{x}(t) = \Big(K + \sum_{i=1}^{n} K_i g(t)(g^{-1}(t-ir) - g^{-1}(t-ir+r))\Big)x(t) + f(t).$$

We define the domain $\Omega = \{(x,t) | t > t_0,\ u(x,t) < 0\}$, where $u(x,t) = (ax+G^{\gamma}(t)b(t))^2 - G^{2(\gamma+\varepsilon)}(t)$. The assumptions of Picard-Lindelöf's theorem are locally satisfied in the domain Ω, therefore throught each $(x,t) \in \Omega$ goes a unique solution of (4). Using the assumptions 1, 2, 3, 4 we compute the trajectory derivative $\dot{u}(x,t)$ along the solution $x(t)$ of (3) on the bound $\partial\Omega$:

$$\dot{u}(x,t) = \frac{2}{g(t)}\{KG^{2(\gamma+\varepsilon)}(t) - G^{2(\gamma+\varepsilon+1)}(t) + G^{2(\gamma+\varepsilon+\lambda)}(t)o(1) \pm$$
$$\pm G^{2\gamma+\varepsilon}[G^{\lambda}(t)b(t)o(1) + KG^{\varepsilon_1}(t)O(1) - \gamma b(t)G(t) + G^{\delta}(t)o(1)]\}.$$

For sufficiently large t the construction of the number ε implies

$$\operatorname{sign} \dot{u}(x,t) = \operatorname{sign} \frac{2a}{g(t)} G^{2(\gamma+\varepsilon)}(t) = 1.$$

Then according to Ważewski's principle [4, p. 282] there is at least one solution $x(t)$ of (4) such that $x(t) \in \Omega$. The asymptotic form of the solution $x(t)$ and also $y(t) = E(t)x(t)$ is obtained from the construction of the domain Ω.

3. Main results

Let the formal solution of equation (1) be expressed in the form (2), where $\varphi(t,C)$ is the general solution of the equation $g(t)\dot{y}(t) = -ay(t)$, consequently $\varphi(t,C) \equiv C\exp\int_{t_0}^{t} \frac{-a}{g(s)}\,ds$, where C is a constant and $f_1(t) = 1$, $f_k(t)$ for $k \geq 2$ are unknown functions for the time being. After substituing

$y(t,C)$ in the equation (1) and comparing coefficients of the same powers $\varphi^k(t,C)$ we obtain an auxiliary system of linear differential equations:

$$(5_k) \qquad g(t)\dot{f}_k(t) = a(k-1)f_k(t) + \sum_{i+j=2}^{N} c_{ij}(t)\sum_{i,j}^{k} \frac{i!j!}{\alpha!\beta!}\mathbf{f}^{\alpha}(t)\mathbf{h}^{\beta}(t)$$

where

$$\mathbf{f}(t) = \{f_k(t)\}_{k=1}^{\infty}, \quad \mathbf{h}(t) = \{h_k(t)\}_{k=1}^{\infty} = \left\{ f_k(t-r)\exp\int_{t-r}^{t}\frac{ak}{g(s)}\,ds \right\}_{k=1}^{\infty}.$$

As $V(\alpha)+V(\beta)=k$ and $|\alpha|+|\beta|\geq 2$ yields $\alpha_l = 0$ and $\beta_l = 0$ for $l \geq k$, the auxiliary system (5_k) is recurrent. Therefore we may define recurrently two sequences of functions:

$$\mathbf{p}(t) = \{p_k(t)\}_{k=1}^{\infty}, \quad \mathbf{q}(t) = \{q_k(t)\}_{k=1}^{\infty} = \left\{ p_k(t-r)\exp\int_{t-r}^{t}\frac{ak}{g(s)}\,ds \right\}_{k=1}^{\infty},$$

$p_1(t) = 1,$

$$p_k(t) = \frac{1}{a(k-1)}\sum_{i+j=2}^{N} c_{ij}(t)\sum_{i,j}^{k}\frac{i!j!}{\alpha!\beta!}\mathbf{p}^{\alpha}(t)\mathbf{q}^{\beta}(t).$$

If $|\beta| \neq 0$, then the expresion $\exp\int_{t-r}^{t}\frac{ak}{g(s)}\,ds$ is included in $\mathbf{q}^{\beta}(t)$ and also in $p_k(t)$. Now using Lemma 2 we describe the asymptotic behaviour of particular solutions of the system 5_k.

THEOREM 1. *Let the functions $p_k(t)$ have the asymptotic form*

$$p_k(t) = E_k(t)G^{\gamma_k}(t)(b_k(t) + O(G^{\varepsilon_k}(t)))$$

as $t\to\infty$ where $\varepsilon_k > 0$, γ_k are constants, $b_k(t)\in C^1[t_k,\infty)$, $b_k(t) = o(g^{\tau}(t))$ as $t\to\infty$ for any positive τ, $g(t)\dot{b}_k(t) = o(g^{\lambda_k}(t))$, as $t\to\infty$, $\lambda_k > 0$ is a constant.

$$E_k(t) = \exp\sum_{i=1}^{n_k} K_k^i \int_{t-ir}^{t-ir+r}\frac{ds}{g(s)}.$$

Assume further there is a sequence $\{\nu_k\}_{k=1}^{\infty}$ such that

$$\nu_k \in (\gamma_k, \gamma_k + \min(\lambda, \delta_k, 1, \varepsilon_k - \Delta_k^*)),$$

where $\Delta_k^* = \max(\Delta_1,\dots,\Delta_{k-1})$, $\Delta_1 = 0$, $\Delta_l = \gamma_l+\varepsilon_l-\nu_l$ for $l = 2,\dots,k-1$.

Then the coeficients $f_k(t)$ of the series (2), which are the solutions of the auxiliary system (5_k), i. e.

(6_k) $f_k(t) =$

$$= \int_t^\infty \frac{-1}{g(s)} \sum_{i+j=2}^{N} c_{ij}(s) \sum_{i,j}^{k} \frac{i!j!}{\alpha!\beta!} \mathbf{f}^\alpha(s)\mathbf{h}^\beta(s) \exp\left\{ - \int_t^s \frac{a(k-1)}{g(u)}\, du \right\} ds$$

can be expressed in the asymptotic form

$$(7_k) \qquad \begin{aligned} f_k(t) &= E_k(t)G^{\gamma_k}(t)\Big(- \frac{b_k(t)}{a(k-1)} + O(g^{\nu_k}(t))\Big) \\ \dot{f}_k(t) &= \frac{1}{g(t)} E_k(t) O(G^{\nu_k}(t)). \end{aligned}$$

Proof. The formulas (6_k) are obtained by integrating the system (5_k). The convergence of (6_k) is evident. It remains to show the asymptotic estimate (7_k). This will be done by induction.

For $k = 2$ the coefficients of the equation (5_2) satisfy the requirements of Lemma 2, thus the solution (6_2) has the form (7_2).

In spite of $\mathbf{f}(t)$ being substituted instead of $\mathbf{p}(t)$ and $\mathbf{h}(t)$ being substituted instead of $\mathbf{q}(t)$, in the recurrent definition of $p_k(t)$, the asymptotic form

$$p_k^*(t) = g^{\gamma_k}(t)(b_{1k}(t) + O(b_{0k}^*(t)g^{\varepsilon_k - \Delta_k^*}(t)))G_k(t)$$

has the same asymptotic properties like $p_k(t)$. Therefore the equation (5_k) satisfies the assumptions of Lemma 2, then (6_k) takes the form (7_k) and the theorem is proved.

Remark. The necessary condition for satisfying assumptions of Theorem 1 is $\lim_{t\to\infty} c_{ij}(t)\exp(-\tau G^{-1}(t)) = 0$. This is satisfied for example if functions $c_{ij}(t)$ have the same asymptotic behaviour $p_k(t)$.

We shall denote

$$y_n(t) = \sum_{k=1}^{n} f_k(t)\varphi^k(t,C) \quad \text{and} \quad \sum_n(t) = \sum_{i+j=2}^{N} c_{ij}(t) \sum_{i,j}^{k} \frac{i!j!}{\alpha!\beta!}\mathbf{f}^\alpha(t)\mathbf{h}^\beta(t).$$

THEOREM 2. *Let the assumptions of Theorem 1 hold and suppose that*

$$\lim_{t\to\infty} f_{n+1}^{-1}(t)\exp(-\tau G^{-1}(t)) = 0,$$

where $\tau < 1$ is a constant. Then for every $C \neq 0$ and $\psi \in C^0[-r,0]$, $\|\psi\| \leq 1$, $\psi(0) = 0$ there exists a solution $y_C(t)$ of equation (1) such that

$$(8) \qquad |y_C(t) - y_n(t)| \leq \delta |f_{n+1}(t)\varphi^{n+1}(t,C)|$$

for $t \in [t_C, \infty)$ where coefficients $f_k(t)$ are the solutions (6_k) of the system (5_k), $\delta > 1$ is a constant, t_C is a function of the parametr C and of δ, n.

Proof. The existence of solution $y_C(t)$ which satisfies the inequality (8) will be proved by Ważewski's principle for retarded functional differential equations $\dot{y} = f(t, y_t)$, where y_t denotes an element of $C^0 = C^0[-r, 0]$ defined as $y_t(\theta) = y(t+\theta)$, $\theta \in [-r, 0]$ for any continuous mapping y from an interval $[-r+t, t]$ into R. For this method see [5]. The function $f(t, y_t) : R \times C^0[-r, 0] \to R$, defined by a formula

$$f(t,\phi) = \frac{1}{g(t)}\Big(- a\phi(0) + \sum_{i+j=2}^{N} c_{ij}(t)\phi^i(0)\phi^j(-r)\Big)$$

is continuous and Lipschitzian in ψ in each compact set in Ω^ε, where

$$\Omega^\varepsilon = \{(t,\psi) | t > t_0 - r;\ \|\varphi - y_{nt}\| < A(t)\},$$

$$\|\phi\| = \sup_{-r\le\theta\le 0} |\phi(\theta)| \quad \text{and} \quad A(t) = (\varepsilon+1) \max_{t-r\le\theta\le t} (f_{n+1}(\theta)\varphi^{n+1}(\theta, C))$$

for a positive constant ε. Thus for any $(t,\phi) \in \Omega^\varepsilon$ there exists the unique solution of the equation $y = f(t, y_t)$ [3,p. 42].

We shall prove that $\omega = \{(y,t) \mid l(y,t) < 0,\ t > t_C\}$, where $l(y,t) = (y - y_n(t))^2 - (\delta f_{n+1}(t)\varphi^{n+1}(t, C))^2$ is the regular polyfacial set with respect to the equation $\dot{y} = f(t, y_t)$, where $f(t, \phi)$ is defined as above. Then for all $K \in (-1, 1)$

$$\begin{aligned}\dot{l}(y,t) = \frac{2}{g(t)}\Big(&(\pm\delta\varphi^{n+1}(t,C)|f_{n+1}(t)|)\Big[- a(y_n(t) \pm \delta\varphi^{n+1}(t,C)|f_{n+1}(t)|)+\\ &+ \sum_{i+j=2}^{N} c_{ij}(t)(y_n(t) \pm \delta|f_{n+1}(t)|\varphi^{n+1}(t,C))^i \times (y_n(t-r)+\\ &+ K\delta|f_{n+1}(t-r)|\varphi^{n+1}(t-r,C))^j + ay_n(t) - \sum_{k=1}^{n}\varphi^k(t,C)\sum_k(t)\Big] -\\ &- (\delta\varphi^{n+1}(t,C))^2[-a(n+1)f_{n+1}^2(t) + g(t)\dot{f}_{n+1}(t)f_{n+1}(t)]\Big).\end{aligned}$$

Using binomial theorem for i, j-power in sumation $\sum_{i+j=2}^{N}$ we obtain

$$\begin{aligned}\dot{l}(y,t) = \frac{2}{g(t)}\Big(&(\delta\varphi^{n+1}(t,C))^2[-a(n+1)f_{n+1}^2(t) - g(t)\dot{f}_{n+1}(t)f_{n+1}(t)]\pm\\ &\pm \delta\varphi^{n+1}(t,C)\Big[- \sum_{k=1}^{n}\varphi^k(t,C)\sum_k(t) + \sum_{i+j=2}^{N} c_{ij}(t)(y_n^i(t)y_n^j(t-r)+\\ +y_n^i(t)\varphi^{n+1}&(t,C)V_1(t) + y_n^j(t-r)\varphi^{n+1}(t,C)V_2(t) + \varphi^{2n+2}(t,C)V_1(t)V_2(t))\Big]\Big),\end{aligned}$$

where

$$V_1(t) = \sum_{l=0}^{j-1} \binom{j}{l} (-1)^{j-l} y_n^l(t-r) \Big(K\delta |f_{n+1}(t-r)| \exp \int_{t-r}^{t} \frac{a(n+1)}{g(s)}\, ds \Big)^{j-l} \times$$
$$\times (\varphi^{n+1}(t,C))^{j-l-1},$$
$$V_2(t) = \sum_{l=0}^{i-1} \binom{i}{l} (-1)^{i-l} y_n^l(t) (\delta |f_{n+1}(t)|)^{i-l} (\varphi^{n+1}(t,C))^{i-l-1}.$$

Therefore

$$\{(\alpha,\beta) \mid V(\alpha)+V(\beta) \le n+1,\ |\alpha|+|\beta| \ge 2,\ \max(\alpha) \le n,\ \max(\beta) \le n\} = \{(\alpha,\beta) | V(\alpha)+V(\beta) \le n+1, |\alpha|+|\beta| \ge 2\},$$

we obtain

$$\sum_{i+j=2}^{N} c_{ij}(t) y_n^i(t) y_n^j(t-r) = \sum_{k=1}^{n=1} \varphi^k(t,C) \sum_k (t) +$$
$$+ \sum_{k=n+2}^{nN} \varphi^k(t,C) \sum_{i+j=2}^{N} c_{ij}(t) \sum_{i_n j_n}^{k} \frac{i!j!}{\alpha!\beta!} \mathbf{f}^\alpha(t) \mathbf{h}^\beta(t),$$

where $\sum_{i_n j_n}^k$ denotes the sumation ower all (α,β) such that $V(\alpha)+V(\beta)=k$, $|\alpha|=i$, $|\beta|=j$, $\max(\alpha)\le n$, $\max(\beta)\le n$.

Eventualy we get

$$\dot{l}(y,t) = \frac{2}{g(t)} \varphi^{2n+2}(t,C) \times$$
$$\times \Big[(anf_{n+1}^2(t) - g(t)\dot{f}_{n+1}(t) f_{n+1}(t))\delta^2 \pm \delta f_{n+1}(t) \sum_{n+1}(t) \Big] \pm$$
$$\pm \delta \varphi^{2n+3}(t,C) \Big[\sum_{k=n+2}^{nN} \varphi^{k-n-2}(t,C) \sum_{i+j=2}^{N} c_{ij}(t) \sum_{i_n j_n}^{k} \frac{i!j!}{\alpha!\beta!} \mathbf{f}^\alpha(t) \mathbf{h}^\beta(t) \times$$
$$\times \sum_{i+j=2}^{N} c_{ij}(t) (+y_n^i(t) V_1(t) + y_n^j(t-r) V_2(t) + \varphi^{n+1}(t,C) V_1(t) V_2(t)) \Big].$$

For sufficiently large $t > t_C$ and $\delta > 1$ we deduce that

$$\operatorname{sign} \dot{l}(y,t) = \operatorname{sign}(anf_{n+1}^2(t) - g(t)\dot{f}_{n+1}(t) f_{n+1}(t)).$$

As $\lim_{t\to\infty} g(t) \frac{\dot{f}_{n+1}(t)}{f_{n+1}(t)} = \lim_{t\to\infty} (G^{\nu_{n+1}-\lambda_{n+1}}(t) = 0$ we obtain

$$\operatorname{sign} \dot{l}(y,t) = \operatorname{sign} anf_{n+1}^2(t) = 1.$$

Consequently ω is the polyfacial set regular with respect to the equation (1), $W = \partial\omega$, $Z = \{(y, t_C) \mid l(y, t_C) \leq 0\}$.

We define $p : B = \overline{Z} \cap (Z \cup W) = Z \to C$ as:

$$p(z)=(y-y_n(t_C))(1-|\psi|)\frac{(f_{n+1}(t)\varphi^{n+1}(t,C))_{t_C}}{r|f_{n+1}(t_C)\varphi^{n+1}(t_C,C)|}+(y_n(t))_{t_C} \text{ for } z=(y,t_C).$$

The mapping $p(z)$ is evidently continuos and for every $z \in B$ $p(z)$ satisfies:

$$(t_C + \theta, p(z)(\theta)) \in \omega \quad \text{for } \theta \in [-r, 0).$$

Moreover it holds: $Z \cap W$ is a retract of W but $Z \cap W$ is not a retract of Z. Then all assumptions of Ważewski principle for retarded functional differential equations are satisfied and thus there exists at least one solution $y_C(t)$ of (1) such that $y_C(t) \in \omega$ for $t > t_C$. The asymptotic form of the solution $y_C(t)$ is obtained from the construction of the domain ω and proof is complete.

Remark. As the relation $h_k(t) = f_k(t-r)\exp\int_{t-r}^{t}\frac{ak}{g(s)}\,ds$ is used in the definition of the sequences $\mathbf{f}(t)$ and $\mathbf{h}(t)$ and the function $h_k(t)$ is used in the definition of $f_{k+1}(t)$ the lefthand end of the existence interval of the function $f_{k+1}(t)$ is greater by r then the lefthand end of the existence interval of $f_k(t)$. If lefthand ends of the existence intervals of the functions $c_{ij}(t)$ are finite then lefthand ends of the existence intervals of the functions $f_k(t)$ must tend to infinity.

Corollary. *If all asumptions of Theorem 2 are satisfied for every n, then there exists the asymptotic expansion of the solution $y_C(t)$ in the form*

$$y_C(t) \approx \sum_{n=1}^{\infty} f_n(t)\varphi^n(t, C),$$

where the coefficients $f_n(t)$ are the solutions (5_n).

Proof. As

$$\lim_{t\to\infty}\frac{f_{n+1}(t)\varphi^{n+1}(t,C)}{f_n(t)\varphi^n(t,C)} =$$

$$= \lim_{t\to\infty} G^{\gamma_{n+1}-\gamma_n}(t)\frac{b_{1n+1}(t)+O(g^{\nu_{n+1}-\gamma_{n+1}}(t))}{b_{1n}(t)+O(g^{\nu_n-\gamma_n}(t))}\varphi(t,C) = 0$$

the assertion is proved.

Example. We consider the equation:

$$\frac{1}{t}\dot{y}(t) = -2y(t) + y^2(t-1) + t\sin t\, y^2(t)y(t-1).$$

In this case we have $a = 2$, $r = 1$, $g(t) = \frac{1}{t}$, $\lambda = 2$, $\psi(t) = -1$. Then the auxiliary system (4_k) have a form:

$$\frac{1}{t}\dot{f}_n(t) = 2(n-1)f_n(t) + \frac{1}{(n-2)!}\exp(a_n t + b_n)(1 + O(t^{-0.9})).$$

Using Lemma 2 we obtain:

$$f_n(t) = \frac{1}{2(n-1)!}\exp(a_n t + b_n)(1 + O(t^{-0.9})),$$

where $a_n = n^2 + 2n - 2$ and $b_n = -\frac{1}{6}(2n^3 + 3n^2 - 11n + 6)$. Then according the Theorem 2 and corollary we obtain

$$y_C(t) \approx \sum_{n=1}^{\infty} \frac{C}{2(n-1)!}\exp\left(a_n t + b_n - \frac{t^2}{2}\right).$$

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COINCIDENCE POINT THEOREMS FOR NON-LINEAR HYBRID CONTRACTIONS IN NON-ARCHIMEDEAN MENGER PROBABILISTIC METRIC SPACES

1. Introduction

In 1942, since K. Menger first introduced the concept of probabilistic metric spaces, the study of these spaces was performed extensively by B. Schweizer and A. Sklar ([21]–[23]) and many authors ([4]–[7], [14], [25], [26], [29]). The theory of probabilistic metric spaces is of fundamental important in probabilistic functional analysis. Recently, a number of fixed point theorems and their applications in probabilistic metric spaces have been proven by several authors: A. T. Bharucha–Reid ([1]), Gh. Bocsan ([2]), G. L. Cain, Jr. and R. H. Kasriel ([3]), S. S. Chang et al. ([8]–[12]), Gh. Constantin ([13]), O. Hadžić ([15], [16]) and M. Stojaković ([27], [28]). In this paper, we introduce the concept of compatibility for single-valued and multi-valued mappings in non-Archimedean Menger probabilistic metric spaces and give some coincidence point theorems for non-linear hybrid contractions, that is, contractive conditions involving single-valued and multi-valued mappings in non-Archimedean Menger probabilistic metric spaces. By using our results, we can also give some common fixed point theorems for single-valued and multi-valued mappings in metric spaces. Our results extend, generalize and improve many results of H. Kaneko and S. Sessa ([17]), S. B. Nadler, Jr.([19]) and many others in metric spaces and probabilistic metric spaces.

2. Preliminaries

Let R denote the set of real numbers and $R^+ = [0, \infty)$. A mapping $\mathcal{F} : R \to R^+$ is called a distribution function, if it is non-decreasing, left-

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continuous with $\inf \mathcal{F} = 0$ and $\sup \mathcal{F} = 1$. We shall denote by $\mathcal{D}$ the set of all distribution functions on R.

DEFINITION 2.1. A commutative, associative and non-decreasing mapping $\triangle : [0,1] \times [0,1] \to [0,1]$ is called a *t-norm*, if $\triangle(a,1) = a$ for all $a \in [0,1]$ and $\triangle(0,0) = 0$.

DEFINITION 2.2. A *Menger probabilistic metric space* (shortly, a *Menger PM-space*) is a triple $(X, \mathcal{F}, \triangle)$, where X is a non-empty set, $\mathcal{F}$ is a mapping from $X \times X$ into $\mathcal{D}$ and $\triangle$ is a t-norm. We shall denote the distribution function $\mathcal{F}(x,y)$ by $F_{x,y}$ and the value of $F_{x,y}$ at $t \in R$ by $F_{x,y}(t)$.

The function $F_{x,y}$ is assumed to satisfy the following conditions:

(MP-1) $F_{x,y}(t) = 1$ for all $t > 0$ if and only if $x = y$,

(MP-2) $F_{x,y}(0) = 0$,

(MP-3) $F_{x,y}(t) = F_{y,x}(t)$ for all $t > 0$,

(MP-4) $F_{x,y}(t_1 + t_2) \geq \triangle(F_{x,y}(t_1), F_{x,y}(t_2))$ for all $x, y \in X$ and for $t, t_1, t_2 > 0$.

DEFINITION 2.3. A Menger PM-space $(X, \mathcal{F}, \triangle)$ is said to be *non-Archimedean*, if the following condition holds:

(MP-5) $F_{x,z}(\max\{t_1, t_2\}) \geq \triangle(F_{x,y}(t_1), F_{y,z}(t_2))$ for all $x, y, z \in X$ and for $t_1, t_2 \geq 0$.

Throughout this paper, H will denote the specific distribution function defined by

$$H(t) = \begin{cases} 0, & t \leq 0, \\ 1, & t > 0. \end{cases}$$

In [21], B. Schweizer and A. Sklar proved that, if $(X, \mathcal{F}, \triangle)$ is a Menger PM-space with the continuous t-norm $\triangle$, then $(X, \mathcal{F}, \triangle)$ is a Hausdorff space in the topology τ induced by the family

$$\{U_x(\varepsilon, \lambda); x \in X, \varepsilon > 0, \lambda \in (0,1)\}$$

of neighborhoods $U_x(\varepsilon, \lambda)$, where

$$U_x(\varepsilon, \lambda) = \{y \in X; F_{x,y}(\varepsilon) > 1 - \lambda\}.$$

Remark 1. In [20], V. Radu proved that, in a Menger PM-space $(X, \mathcal{F}, \triangle)$ with the t-norm $\triangle(a,b) \geq \triangle_m(a,b) = \max\{a + b - 1, 0\}$ for all $a, b \in [0,1]$, the topology τ induced by the family $\{U_x(\varepsilon, \lambda)\}$ of neighborhoods is equivalent to the topology induced by the metric d, where the metric d is defined by

$$d(x,y) = \sup\{t \in [0,1] : F_{x,y}(t) < 1 - t\}$$

for all $x, y \in X$.

Let $\mathcal{G}$ be the family of functions $g : [0,1] \to [0,\infty)$ such that each g is continuous, strictly decreasing and $g(1) = 0$.

DEFINITION 2.4. A Menger PM-space $(X, \mathcal{F}, \triangle)$ is said to be of *type* $(C)_g$, if there exists a point $g \in G$ such that

$$g(F_{x,z}(t)) \leq g(F_{x,y}(t)) + g(F_{y,z}(t))$$

for all $x, y, z \in X$ and $t \geq 0$.

DEFINITION 2.5. A non-Archimedean Menger PM-space $(X, \mathcal{F}, \triangle)$ is said to be of *type* $(D)_g$, if there exists a point $g \in \mathcal{G}$ such that

$$g(\triangle(s,t)) \leq g(s) + g(t)$$

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for all $s, t \in [0,1]$.

Remark 2. (1) If a non-Archimedean Menger PM-space $(X, \mathcal{F}, \triangle)$ is of type $(D)_g$, then it is of type $(C)_g$.

(2) If $(X, \mathcal{F}, \triangle)$ is a Menger PM-space with the t-norm

$$\triangle(a,b) \geq \triangle_m(a,b) \quad \text{for all } a, b \in [0,1],$$

then it is of type $(D)_g$ for $g \in G$ defined by $g(t) = 1 - t$ for all $t \in [0,1]$.

DEFINITION 2.6. Let $(X, \mathcal{F}, \triangle)$ be a Menger PM-space with the continuous t-norm $\triangle$. A subset A of X is said to be *probabilistically bounded*, if $\sup_{t>0} D_A(t) = 1$, where

$$D_A(t) = \sup_{s<t} \inf_{p,g \in A} F_{p,q}(s),$$

$D_A(t)$ is called the probabilistic diameter of A.

Remark 3. ([5]) Let $(X, \mathcal{F}, \triangle)$ be a Menger PM-space with the continuous t-norm $\triangle$.

(1) If A is a probabilistically bounded subset of X, then $D_A(t)$ is a distribution function.

(2) If A and B are probabilistically bounded subsets of X, then $A \cup B$ is also probabilistically bounded.

Let $(X, \mathcal{F}, \triangle)$ be a Menger PM-space with the continuous t-norm $\triangle$ and Ω be the family of all non-empty τ-closed and probabilistically bounded subsets of X. We define a mapping $\widetilde{\mathcal{F}} : \Omega \times \Omega \to \mathcal{D}$ as follows:

$$\widetilde{F}_{A,B}(t) = \sup_{s<t} \triangle\{\inf_{a\in A} \sup_{b\in B} F_{a,b}(s), \inf_{b\in B} \sup_{a\in A} F_{a,b}(s)\}$$

for all $A, B \in \Omega$, where we denote $\widetilde{\mathcal{F}}(A,B)$ by $\widetilde{F}_{A,B}$ and the value of $\widetilde{F}_{A,B}$ at $t \in R$ by $\widetilde{F}_{A,B}(t)$. $\widetilde{\mathcal{F}}$ is called the Menger–Hausdorff metric induced by $\mathcal{F}$.

Let $A \in \Omega$ and $x \in X$. The probabilistic distance between the point x and the set A is the function $F_{x,A}$ defined by

$$F_{x,A}(t) = \sup_{s<t} \sup_{y \in A} F_{x,y}(s)$$

for all $t \geq 0$.

LEMMA 2.1. ([5]) (1) $(\Omega, \widetilde{\mathcal{F}}, \triangle)$ *is a Menger PM-space.*
(2) $F_{x,A}(t) = 1$ *for all* $t > 0$ *if and only if* $x \in A$.
(3) $F_{x,A}(t_1 + t_2) \geq \triangle(F_{x,y}(t_1), F_{y,A}(t_2))$ *for all* $t_1, t_2 \geq 0$.
(4) *For any* $A, B \in \Omega$ *and* $x \in A$, $F_{x,B}(t) \geq \widetilde{F}_{A,B}(t)$ *for all* $t \geq 0$.

Let (X, d) be a metric space and $CB(X)$ be the family of all non-empty closed and bounded subsets of X. Let δ be the Hausdorff metric on $CB(X)$ induced be the metric d, that is,

$$\delta(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{x \in B} d(x, A)\}$$

for all $A, B \in CB(X)$, where $d(x, A) = \inf_{y \in A} d(x, y)$.

Remark 4. (1) $(CB(X), \delta)$ is a metric space.
(2) If (X, d) is complete, then $(CB(X), \delta)$ is also complete ([18]).

Remark 5. (1) Let (X, d) be a complete metric space. If we define $\mathcal{F} : X \times X \to \mathcal{D}$ as follows:

$$\mathcal{F}(x, y)(t) = F_{x,y}(t) = H(t - d(x, y))$$

for all $x, y \in X$ and $t \in R$, then the space $(X, \mathcal{F}, \triangle)$ with the t-norm $\triangle(a, b) = \min\{a, b\}$ for all $a, b \in [0, 1]$ is a τ-complete Menger PM-space ([24]). Conversely, if $(X, \mathcal{F}, \triangle)$ is a τ-complete Menger PM-space with the t-norm $\triangle(a, b) = \min\{a, b\} \geq \triangle_m(a, b)$ for all $a, b \in [0, 1]$, by Theorem 3.1 in [6], then (X, d) is a d-complete metric space, where the metric d is defined by

$$d(x, y) = \sup\{t \in [0, 1]; F_{x,y}(t) < 1 - t\}$$

for all $x, y \in X$.

(2) Let $A \in CB(X)$ and $x \in X$. We define the probabilistic distance $F_{x,A}$ between the point x and the set A as follows:

$$F_{x,A}(t) = H(t - d(x, A))$$

for all $t \in R$ ([5]).

(3) If we define $\widetilde{\mathcal{F}} : CB(X) \times CB(X) \to \mathcal{D}$ by

$$\widetilde{\mathcal{F}}(A, B)(t) = \widetilde{F}_{A,B}(t) = H(t - \delta(A, B))$$

for all $A, B \in CB(X)$ and $t \in R$, then $\widetilde{\mathcal{F}}$ is the Menger–Hausdorff metric

induced by $\mathcal{F}$ ([5]). Thus we can show that, if $(X, \mathcal{F}, \triangle)$ is a τ-complete Menger PM-space with the t-norm $\triangle(a,b) \geq \triangle_m(a,b)$ for all $a, b \in [0,1]$, then $(\Omega, \widetilde{\mathcal{F}}, \triangle)$ is also a τ-complete Menger PM-space.

3. Coincidence point theorems (I)

Throughout this section, let $(X, \mathcal{F}, \triangle)$ be a τ-complete non-Archimedean Menger PM-space of type $(D)_g$ with the continuous t-norm $\triangle(a,b) = \min\{a,b\}$ for all $a, b \in [0,1]$. Let Φ be the family of mappings $\phi : (R^+)^5 \to R^+$ such that each ϕ is non-decreasing for each variable, right-continuous and for any $t \geq 0$,

$$\phi(t,t,t,2t,0) \leq \psi(t),$$

where the function $\psi : R^+ \to R^+$ is non-decreasing, right-continuous and $\psi^n(t) \to 0$, as $n \to \infty$, for all $t > 0$.

LEMMA 3.1. ([9]) *Let $\psi : R^+ \to R^+$ be non-decreasing, right-continuous and $\psi^n(t) \to 0$, as $n \to \infty$, for all $t > 0$. Then we have the following assertions*:

(1) *$\psi(t) < t$ for all $t > 0$.*

(2) *If $t \leq \psi(t)$, then $t = 0$.*

Let f be a mapping from X into itself and T be a multi-valued mapping from X into Ω.

DEFINITION 3.1. The mappings f and T are said to be *commuting*, if $fTx \in \Omega$ and $fTx = Tfx$ for all $x \in X$.

DEFINITION 3.2. The mappings f and T are said to be *compatible*, if $fTx \in \Omega$ and

$$\lim_{n\to\infty} g(\widetilde{F}_{fTx_n, Tfx_n}(t)) = 0$$

for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Tx_n = A \in \Omega$ and $\lim_{n\to\infty} fx_n = z \in A$, where $g \in G$.

Remark 6. By Definitions 3.1 and 3.2, any commuting mappings are compatible but the converse is not true.

Now, we are ready to give our main theorems.

THEOREM 3.2. *Let f be a τ-continuous mapping from X into itself and $\{T_n\}_{n=1}^{\infty}$ be a sequence of τ-continuous multi-valued mappings from X into Ω satisfying the following conditions*:

(3.1) $T_n(X) \subset f(X)$ *for* $n = 1, 2, \ldots,$

(3.2) f *and* T_n *are compatible for* $n = 1, 2, \ldots,$

$$(3.3)\quad g(F_{T_ix,T_jy}(t)) \le \phi(g(F_{fx,fy}(t)), g(F_{fx,T_ix}(t)), g(F_{fy,T_jy}(t)), g(F_{fx,T_jy}(t)), g(F_{fy,T_ix}(t)))$$

for all $x, y \in X$, $t \ge 0$ *and* $i \ne j$, $i, j = 1, 2, \ldots$, *where* $g \in \mathcal{G}$ *and* $\phi \in \Phi$. *Suppose further that*

(3.4) *for any* $x \in X$ *and* $a \in T_n x$, $n = 1, 2, \ldots$, *there exists a point* $b \in T_{n+1}a$ *such that*

$$F_{a,b}(t) \ge \widetilde{F}_{T_n x, T_{n+1} a}(t)$$

for all $t > 0$.

Then there exists a point $z \in X$ *such that* $fz \in T_n z$ *for* $n = 1, 2, \ldots$, *that is,* z *is a coincidence point of* f *and* T_n.

Proof. Since $T_n(X) \subset f(X)$ for $n = 1, 2, \ldots$, by (3.4), and $g \in \mathcal{G}$, for an arbitrary point $x_0 \in X$, we can choose a point x_1 in X such that $fx_1 \in T_1 x_0 \in \Omega$. For this point x_1, there exists a point x_2 in X such that $fx_2 \in T_2 x_1 \in \Omega$ and

$$g(F_{fx_1,fx_2}(t)) \le g(\widetilde{F}_{T_1x_0,T_2x_1}(t))$$

for all $t \ge 0$. Similarly, there exists a point $x_3 \in X$ such that $fx_3 \in T_3 x_2 \in \Omega$ and

$$g(F_{fx_2,fx_3}(t)) \le g(\widetilde{F}_{T_2x_1,T_3x_2}(t))$$

for all $t \ge 0$. Inductively, we can obtain a sequence $\{x_n\}$ in X such that $fx_n \in T_n x_{n-1} \in \Omega$ and

$$g(F_{fx_n,fx_{n+1}}(t)) \le g(\widetilde{F}_{T_nx_{n-1},T_{n+1}x_n}(t))$$

for all $t \ge 0$. Now, we shall prove that the sequence $\{fx_n\}$ is a Cauchy sequence in X. In fact, for $n = 1, 2, \ldots$, by Lemma 2.1 (4) and by (3.3), (3.4); since $g \in \mathcal{G}$, we have

$$\begin{aligned}
(3.5)\quad g(F_{fx_n,fx_{n+1}}(t)) &\le g(\widetilde{F}_{T_nx_{n-1},T_{n+1}x_n}(t)) \\
&\le \phi(g(F_{fx_{n-1},fx_n}(t)), g(F_{fx_{n-1},T_nx_{n-1}}(t)), \\
&\qquad g(F_{fx_n,T_{n+1}x_n}(t)), g(F_{fx_{n-1},T_{n+1}x_n}(t)), \\
&\qquad g(F_{fx_n,T_nx_{n-1}}(t))) \\
&\le \phi(g(F_{fx_{n-1},fx_n}(t)), g(F_{fx_{n-1},fx_n}(t)), \\
&\qquad g(F_{fx_n,fx_{n+1}}(t)), g(F_{fx_{n-1},fx_{n+1}}(t)), \\
&\qquad g(F_{fx_n,fx_n}(t))) \\
&\le \phi(g(F_{fx_{n-1},fx_n}(t)), g(F_{fx_{n-1},fx_n}(t)), \\
&\qquad g(F_{fx_n,fx_{n+1}}(t)), g(F_{fx_n,fx_{n+1}}(t))+ \\
&\qquad + g(F_{fx_{n-1},fx_n}(t)), 0).
\end{aligned}$$

If $g(F_{fx_{n-1},fx_n}(t_0) < g(F_{fx_n,fx_{n+1}}(t_0))$ for some $t_0 > 0$, from (3.5) and Lemma 3.1 (1) it follows that

$$\begin{aligned} g(F_{fx_n,fx_{n+1}}(t)) &\le \phi(g(F_{fx_n,fx_{n+1}}(t_0)), g(F_{fx_{n+1},fx_n}(t_0)), \\ &\qquad g(F_{fx_n,fx_{n+1}}(t_0)), 2g(F_{fx_n,fx_{n+1}}(t_0)), 0) \\ &\le \psi(g(F_{fx_n,fx_{n+1}}(t_0)) \\ &< g(F_{fx_n,fx_{n+1}}(t_0)), \end{aligned}$$

which is a contradiction. Thus, for any $t > 0$, we have

$$g(F_{fx_n,fx_{n+1}}(t)) \le g(F_{fx_{n-1},fx_n}(t_0))$$

for $n = 1, 2, \ldots$, and so, by (3.5),

$$\begin{aligned} (3.6) \qquad g(F_{fx_n,fx_{n+1}}(t)) &\le \phi(g(F_{fx_{n-1},fx_n}(t)), g(F_{fx_{n-1},fx_n}(t)), \\ &\qquad g(F_{fx_{n-1},fx_n}, 2g(F_{fx_{n-1},fx_n}(t)), 0) \\ &\le \psi(g(F_{fx_{n-1},fx_n}(t)) \\ &\le \ldots \\ &\le \psi^n(g(F_{fx_0,fx_1}(t))) \end{aligned}$$

for all $t > 0$. Hence, for any positive integers m, n with $m > n$ and $t > 0$,

$$\begin{aligned} g(F_{fx_n,fx_{n+m}}(t)) &\le g(F_{fx_n,fx_{n+1}}(t)) + \ldots + g(F_{fx_{n+m-1},fx_{n+m}}(t)) \\ &\le \sum_{i=1}^{n+m-1} \psi^i g(F_{fx_0,fx_1}(t)), \\ &\to 0 \end{aligned}$$

as $n \to \infty$, which implies that $F_{fx_n,fx_{n+m}}(t) \to 1$, as $n \to \infty$, for any positive integer m, that is, $\{fx_n\}$ is a Cauchy sequence in X. Since $(X, \mathcal{F}, \triangle)$ is τ-complete, the sequence $\{fx_n\}$ converges to a point $z \in X$. On the other hand, by (3.4) and (3.6), since we have

$$g(F_{fx_n,fx_{n+m}}(t)) \le g(\widetilde{F}_{T_n x_{n-1},T_{n+1}x_n} \le \psi^n(g(F_{fx_0,fx_1}(t))),$$

letting $n \to \infty$, we have $g(\widetilde{F}_{T_n x_{n-1},T_{n+1}x_n}(t)) \to 0$, which implies

$$\widetilde{F}_{T_n x_{n-1},T_{n+m}x_{n+m-1}}(t) \to 1,$$

as $n \to \infty$, for all $t > 0$, that is, $\{T_n x_{n-1}\}$ is a Cauchy sequence in $(\Omega, \widetilde{\mathcal{F}}, \triangle)$ and so, by Remark 5 (3), since $(\Omega, \widetilde{\mathcal{F}}, \triangle)$ is τ-complete, the sequence $\{T_n x_{n-1}\}$ converges to a set A in Ω. Next, we shall prove that $z \in A$.

Indeed, we have

$$\begin{aligned}g(F_{z,A}(t)) &\le g(F_{z,fx_n}(t)) + g(F_{fx_n,T_nx_{n-1}}(t)) + g(\widetilde{F}_{T_nx_{n-1},A}(t))\\ &\le g(F_{z,fx_n}(t)) + g(F_{fx_n,fx_n}(t)) + g(\widetilde{F}_{T_nx_{n-1},A}(t)),\\ &\to 0\end{aligned}$$

as $n \to \infty$, which implies $F_{z,A}(t) \to 1$, as $n \to \infty$, for all $t > 0$. Thus, since $A \in \Omega$, $z \in A$. Therefore, since f and T_n are compatible for $n = 1, 2, \ldots$, we have

$$\begin{aligned}g(F_{fz,T_nz}(t)) &\le g(F_{fz,ffx_n}(t)) + g(F_{ffx_n,T_nz}(t))\\ &\le g(F_{fz,ffx_n}(t)) + g(\widetilde{F}_{fT_nx_{n-1},T_nz}(t))\\ &\le g(F_{fz,ffx_n}(t)) + g(\widetilde{F}_{fT_nx_{n-1},T_nfx_{n-1}}(t))\\ &\quad + g(\widetilde{F}_{T_nfx_{n-1},T_nz}(t)),\\ &\to 0\end{aligned}$$

as $n \to \infty$, that is, $F_{fz,T_nz}(t) \to 1$, as $n \to \infty$. Since $T_nz \in \Omega$, we have $fz \in T_nz$ for $n = 1, 2, \ldots$. This completes the proof.

In Theorem 3.2, taking $T_1 = S$, $T_2 = T$, $T_3 = S$, $T_4 = T, \ldots$, we have the following result.

COROLLARY 3.3. *Let f be a τ-continuous mapping from X into itself and S, T be τ-continuous multi-valued mappings from X into Ω satisfying the following conditions*:

(3.7) $S(X) \cup T(X) \subset f(X)$,

(3.8) *the pairs f, S and f, T are compatible*,

(3.9) $$g(F_{Sx,Ty}(t)) \le \phi(g(F_{fx,fy}(t)), g(F_{fx,Sx}(t)), g(F_{fy,Ty}(t)), g(F_{fx,Ty}(t)), g(F_{fy,Sx}(t)))$$

for all $x, y \in X$ and $t \ge 0$, where $g \in \mathcal{G}$ and $\phi \in \Phi$.

Suppose further that

(3.10) *for any $x \in X$ and $a \in Sx$, there exists a point $b \in Ta$ such that*

$$\widetilde{F}_{a,b}(t) \ge F_{Sx,Ta}(t) \quad \textit{for all } t > 0.$$

Then there exists a point $z \in X$ such that $fz \in Sz \cap Tz$, that is, z is a coincidence point of the pairs f, S and f, T.

By Theorem 3.2 with $f = id_X$(: the identity mapping on X), we have the following result.

COROLLARY 3.4. *Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of τ-continuous multi-valued mappings from X into Ω satisfying the conditions* (3.4) *and*

$$(3.11)\quad g(\widetilde{F}_{T_ix,T_jy}(t)) \leq \phi(g(F_{x,y}(t)), g(F_{x,T_ix}(t)), g(F_{y,T_jy}(t)), g(F_{x,T_jy}(t)), g(F_{y,T_ix}(t)))$$

for all $x, y, z \in X$ and $t \geq 0$, where $g \in \mathcal{G}$ and $\phi \in \Phi$. Then there exists a point $z \in X$ such that $z \in T_n z$ for $n = 1, 2, \ldots$, that is, the point z is a common fixed point of T_n.

Remark 7. In Theorem 3.2, if we replace the condition (3.3) by the following condition:

$$(3.12)\quad g(\widetilde{F}_{T_ix,T_jy}(t)) \leq \psi(\max\{g(F_{fx,fy}(t)), g(F_{fx,T_ix}(t)), g(F_{fy,T_jy}(t)), g(F_{fx,T_jy}(t)), g(F_{fy,T_ix}(t))\})$$

for all $x, y \in X$ and $t \geq 0$, where $g \in \mathcal{G}$, then the conclusion of Theorem 3.2 is still true.

Let $(X, \mathcal{F}, \triangle)$ be a τ-complete Menger PM-space with the continuous t-norm $\triangle(a, b) \geq \triangle_m(a, b)$ for all $a, b \in [0, 1]$.

THEOREM 3.5. *Let f be a τ-continuous mapping from X into itself and $\{T_n\}_{n=1}^{\infty}$ be a sequence of τ-continuous multi-valued mappings from X into Ω satisfying the conditions* (3.1), (3.2), (3.4) *and*

(3.13) *there exists a constant $k > 1$ such that*

$$\widetilde{F}_{T_ix,T_jy}(t) \geq \min\{F_{fx,fy}(kt), F_{fx,T_ix}(kt), F_{fy,T_jy}(kt), F_{fx,T_jy}(kt), F_{fy,T_ix}(kt)\}$$

for all $x, y \in X$ and $t \geq 0$.

Then there exists a point $z \in X$ such that $fz \in T_n z$ for $n = 1, 2, \ldots$.

Proof. By using Remarks 2 (2) and 6 with $\psi(t) = ct$ for some $c \in (0, 1)$ and $t \geq 0$ and Theorem 3.2 with $g \in \mathcal{G}$ defined by $g(t) = 1 - t$ for all $t \in [0, 1]$, this theorem follows immediately.

Remark 8. In Theorems 3.2 and 3.5 and Corollaries 3.3 and 3.4, even though the condition of the compatibility is replaced by the commutativity, the theorems and corollaries are still true.

4. Coincidence point theorems (II)

In this section, we give some coincidence point theorems for a single-valued mapping and multi-valued mappings in metric spaces by using results

in the previous section. Our results extend some results of H. Kaneko and S. Sessa ([17]) and S. B. Nadler, Jr. ([19]) and others.

DEFINITION 4.1. A metric space (X, d) is said to be *non-Archimedean*, if the following condition holds

$$d(x,y) \leq \max\{d(x,z), d(z,y)\}$$

for all $x, y, z \in X$.

Throughout this section, let (X, d) be a non-Archimedean metric space and $(CB(X), \delta)$ be the Hausdorff metric space induced by the metric d. Let f be a mapping from X into itself and T be a multi-valued mapping from X into $CB(X)$.

DEFINITION 4.2. The mappings f and T are said to be *commuting*, if $fTx \in CB(X)$ and $fTx = Tfx$ for all $x \in X$.

DEFINITION 4.3. ([17]) The mappings f and T are said to be *compatible*, if $fTx \in \Omega$ for all $x \in X$ and

$$\lim_{n\to\infty} H(fTx_n, Tfx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Tx_n = A \in CB(X)$ and $\lim_{n\to\infty} fx_n = z \in A$.

Remark 9. From Definitions 4.2 and 4.3, any commuting mappings are compatible but the converse is not true.

EXAMPLE. ([17]) Let $X = [1, \infty)$ with the Euclidean metric d. Define $f : X \to X$ and $T : X \to CB(X)$ by

$$f(x) = 2x^4 - 1 \quad \text{and} \quad Tx = [1, x^2],$$

for all $x \geq 1$, respectively. Then the mappings f and T are compatible but not commuting.

Now, we give our main theorems.

THEOREM 4.1. *Let (X, d) be a complete non-Archimedean metric space and $C\ (X)$ be the family of all non-empty compact subsets of X. Let f be a continuous mapping from X into itself and $\{T_n\}_{n=1}^{\infty}$ be a sequence of continuous multi-valued mappings from X into $C(X)$ satisfying the following conditions*:

(4.1) $T_n(X) \subset f(X)$ *for* $n = 1, 2, \ldots,$

(4.2) *f and T_n are compatible for $n = 1, 2, \ldots$,*

(4.3) *there exists an $\alpha \in (0,1)$ such that*

$$\delta(T_i x, T_j y) \le \alpha \max\{d(fx, fy), d(fx, T_i x), d(fy, T_j y), d(fx, T_j y), d(fy, T_i x)\}$$

for all $x, y \in X$ and $i \neq j$, $i, j, = 1, 2, \ldots$.

Then there exists a point $z \in X$ such that $fz \in T_n z$ for $n = 1, 2, \ldots$, that is, z is a coincidence point of f and f_n.

Proof. By Remark 5 and (4.3), we have

$$\begin{aligned}\widetilde{F}_{T_i x, T_j y}(t) &= H(t - \delta(T_i x, T_j y)) \\ &\ge H(t - \alpha \max\{d(fx, fy), d(fx, T_i x), d(fy, T_j y), \\ &\quad d(fx, T_j y), d(fy, T_i x)\}) \\ &= H\left(\frac{t}{\alpha} - \max\{d(fx, fy), d(fx, T_i x), d(fy, T_j y), \right. \\ &\quad \left. d(fx, T_j y), d(fy, T_i x)\}\right) \\ &= \min\left\{F_{fx,fy}\left(\frac{t}{\alpha}\right), F_{fx,T_i x}\left(\frac{t}{\alpha}\right), F_{fy,T_j y}\left(\frac{t}{\alpha}\right), \right. \\ &\quad \left. F_{fx,T_j x}\left(\frac{t}{\alpha}\right), F_{fy,T_i x}\left(\frac{t}{\alpha}\right)\right\}\end{aligned}$$

for all $x, y \in X$, $t \ge 0$ and $i \neq j$, $i, j = 1, 2, \ldots$.

Moreover, by S. B. Nadler ([19]), for any $x \in X$ and $a \in T_n x$ for $n = = 1, 2, \ldots$, there exists a point $b \in T_{n+1} a$ such that

$$d(a, b) \le \delta(T_n x, T_{n+1} a)$$

and so we have

$$\begin{aligned}F_{a,b}(t) &= H(t - d(a, b)) \\ &\ge H(t - \delta(T_n x, T_{n+1} a)) \\ &= \widetilde{F}_{T_n x, T_{n+1} a}(t)\end{aligned}$$

for all $t \ge 0$. Therefore, all the conditions of Theorem 3.5 are satisfied and hence this theorem follows immediately. This completes the proof.

Remark 10. Theorem 4.1 extend Theorems 2 and 4 of H. Kaneko and S. Sessa ([17]).

COROLLARY 4.2. *Let (X, d) be a complete non-Archimedean metric space. Let f be a continuous mapping from X into itself and T be a multi-*

valued mapping from X *into* $C(X)$ *such that:*

(4.4) $T(X) \subset f(X)$,

(4.5) f *and* T *are commuting,*

(4.6) *there exists an* $h \in [0,1)$ *such that*

$$\delta(Tx, Ty) \leq hd(fx, fy), \quad \text{for all } x, y \in X.$$

Then there exists a point $z \in X$ *such that* $fz \in Tz$.

LEMMA 4.3. ([17]) *Let* (X, d) *be a metric space. Let* f *be a mapping from* X *into itself and* T *be a multi-valued mapping from* X *into* $C(X)$ *such that the mappings* f *and* T *are compatible. If* $fz \in Tz$ *for some* $z \in X$, *then* $fTz = Tfz$.

By using Theorem 4.1 with $T_n = T$, $n = 1, 2, \ldots$, we have the following one.

THEOREM 4.4. *Let* (X, d) *and* $C(X)$ *be as in Theorem* 4.1. *Let* f *be a continuous mapping from* X *into itself and* T *be a continuous multi-valued mapping from* X *into* $C(X)$ *satisfying the following conditions*:

(4.7) $T(X) \subset f(X)$,

(4.8) f *and* T *are compatible,*

(4.9) *there exists an* $\alpha \in (0,1)$ *such that*

$$\begin{aligned}\delta(Tx, Ty) \leq \alpha \max\{&d(fx, fy), d(fx, Tx), d(fy, Ty),\\ &d(fx, Ty), d(fy, Tx)\}, \quad \text{for all } x, y \in X.\end{aligned}$$

Suppose also that, for each $x \in X$ *either* $fx \neq f^2x$ *implies* $fx \notin Tx$ *or* $fx \in Tx$ *implies* $\lim_{n\to\infty} f^n x = z$ *for some* $z \in X$. *Then* f *and* T *have a common fixed point in* X.

Proof. By Lemma 4.3 and the proof of Theorem 3 of H. Kaneko and S. Sessa ([17]), this theorem follows immediately.

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A NOTE ON OPIAL TYPE INEQUALITIES INVOLVING PARTIAL SUMS

1. Introduction

In [8] Z. Opial found an interesting and useful integral inequality involving a function and its derivative. Further, a large number of papers deal with various extensions and generalizations of the Opial inequality (see [1], [5]–[10]). The main purpose of the present note is to establish two new Opial type inequalities involving partial sums. Our results are based on the Hardy inequality involving partial sums (see [2]–[4]) and the discrete analogue of the Opial inequality given by Wong in [10].

2. Result

First, we recall the known inequalities.

LEMMA 1 (see [2]–[4]). *Let* $\lambda_n > 0$, $a_n \geq 0$, $n = 1, 2, \ldots$, *and* $\Lambda_n = \lambda_1 + \ldots + \lambda_n$, $A_n = \lambda_1 a_1 + \ldots + \lambda_n a_n$. *Then*

$$\sum_{n=1}^{m} \lambda_n \left(\frac{A_n}{\Lambda_n}\right)^p \leq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{m} \lambda_n a_n^p, \qquad p > 1.$$

LEMMA 2 (see [10]). *For nondecreasing sequence of nonnegative numbers* $\{u_n\}_1^\infty$ *we have*

$$\sum_{n=1}^{m} u_n^p (u_n - u_{n-1}) \leq \frac{(m+1)^p}{p+1} \sum_{n=1}^{m} (u_n - n_{n-1})^{p+1}, \qquad p \geq 1,$$

with $u_0 = 0$.

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Our result is given in the following theorem.

THEOREM. *Let $p \geq 1$, $q \geq 1$ and λ_n, a_n, Λ_n, A_n be as defined in Lemma 1. Then*

$$\sum_{n=1}^{m} \frac{A_n^p(A_n^q - A_{n-1}^q)}{\Lambda_n^{p+q-1}} \leq q(\frac{p+q}{p+q-1})^{p+q-1} \sum_{n=1}^{m} \lambda_n a_n^{p+q}, \tag{1}$$

$$\sum_{n=1}^{m} A_n^p(A_n^q - A_{n-1}^q) \leq \frac{q(m+1)^{p+q-1}}{p+q} \sum_{n=1}^{m} (\lambda_n a_n)^{p+q}, \tag{2}$$

where any number with suffix zero is zero.

Proof. Since $A_{n-1} \leq A_n$, we have

$$\begin{aligned} A_n^p(A_n^q - A_{n-1}^q) &= A_n^p\Big(\sum_{k=0}^{q-1} A_n^{q-1-k} A_{n-1}^k\Big)(A_n - A_{n-1}) \leq \\ &\leq qA_n^{p+q-1}(A_n - A_{n-1}) \end{aligned} \tag{3}$$

which implies, by using the Hölder inequality with indices $p+q$, $\frac{p+q}{p+q-1}$ and Lemma 1, the inequality

$$\begin{aligned} \sum_{n-1}^{m} \frac{A_n^p(A_n^q - A_{n-1}^q)}{\Lambda^{p+q-1}} &\leq \\ &\leq q\sum_{n=1}^{m}\left(\frac{A_n}{\Lambda_n}\right)^{p+q-1} (\lambda_n a_n) = \\ &= q\sum_{n=1}^{m} \alpha_n^{\frac{1}{p+q}} a_n \lambda_n^{\frac{p+q-1}{p+q}} \left(\frac{A_n}{\Lambda_n}\right)^{p+q-1} \leq \\ &\leq q\Big(\sum_{n=1}^{m} \lambda_n a_n^{p+q}\Big)^{\frac{1}{p+q}} \Big[\sum_{n=1}^{m} \lambda_n (\frac{A_n}{\Lambda_n})^{p+q}\Big]^{\frac{p+q-1}{p+q}} \leq \\ &\leq q\Big(\sum_{n=1}^{m} \lambda_n a_n^{p+q}\Big)^{\frac{1}{p+q}} \Big[\Big(\frac{p+q}{p+q-1}\Big)^{p+q} \sum_{n=1}^{m} \lambda_n a_n^{p+q}\Big]^{\frac{p+q-1}{p+q}} = \\ &= q\Big(\frac{p+q}{p+q-1}\Big)^{p+q-1} \sum_{n=1}^{m} \lambda_n a_n^{p+q} \end{aligned}$$

proving (1).

From (3), using Lemma 2, we have

$$\sum_{n=1}^{m} A_n^p (A_n^q - A_{n-1}^q) \le q \sum_{n=1}^{m} A_n^{p+q-1}(A_n - A_{n-1}) \le$$

$$\le \frac{q(m+1)^{p+q-1}}{p+q} \sum_{n=1}^{m} (A_n - A_{n-1})^{p+q} =$$

$$= \frac{q(m+1)^{p+q-1}}{p+q} \sum_{n=1}^{m} (\lambda_n a_n)^{p+q}$$

which proves (2).

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VARIETIES OF ALGEBRAS HAVING A DISTRIBUTIVE LATTICE OF SUBVARIETIES

1. Introduction and preliminaries

The question of describing in term of identities the varieties having a distributive lattice of subvarieties was raised by L. Bokut in 1976 in [5, problem 19] and in [3] too. Since the survey [2] of V.A. Artamonov in 1978 many results concerning the topic have been obtained [1, 11, 7, 9, 8, 14, 12, 13].

In the paper we consider the absolutely free algebra $F = K\{X\}$ of infinite rank on a countable set X of free generators $x_1, x_2, \ldots$ over a fixed field K of characteristic zero. F_m is the subalgebra of rank m generated by $x_1, x_2, \ldots, x_m$. We denote by S_n and GL_m the symmetric group and the general linear group, acting on the set of symbols $\{1, 2, \ldots, n\}$ and on a m-dimensional vector space, respectively.

Let I be a T-ideal in F and $\mathfrak{M}$ a variety corresponding to I. We denote F/I by $F\{\mathfrak{M}\}$ and $F_m/F_m \cap I$ by $F_m(\mathfrak{M})$. The space $P_n(\mathfrak{M})$ of all multilinear polynomials of degree n from $F_n(\mathfrak{M})$ has a structure of a left S_n-module. $F_m(\mathfrak{M})$ is a left GL_m-module too.

The irreducible S_n- and GL_m-modules are described by Young diagrams. For a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ of n, $\lambda_1 \geq \ldots \geq \lambda_r \geq 0$, $\lambda_1 + \ldots + \lambda_r = n$ we denote by $M(\lambda)$ and $N_m(\lambda)$ the S_n- and GL_m-modules corresponding to λ. One can look into [4, 15, 6, 10] for details on the representation theory of the symmetric and general linear groups.

The subvarieties of $\mathfrak{M}$ form a lattice $\Lambda(\mathfrak{M})$ with respect to the intersection and the union of subvarieties. The question of distributivity has been treated by consideration of $P_n(\mathfrak{M})$. Because of [2] $\Lambda(\mathfrak{M})$ is distributive iff $P_n(\mathfrak{M})$ for all n is a sum of pairwise non-isomorphic irreducible S_n-modules $M(\lambda)$.

It is known [6] that the homogeneous component $F_m^{(n)}(\mathfrak{M})$ of $F_m(\mathfrak{M})$ and $P_n(\mathfrak{M})$ have the same module structure. If $P_n(\mathfrak{M}) = \sum_\lambda k(\lambda)M(\lambda)$, then $F_m^{(n)}(\mathfrak{M}) = \sum_\lambda k(\lambda)N_m(\lambda)$. Thus for convenience the investigations in the paper are on the GL_m-structure of $F_m^{(n)}$. For $n = 2,3$ we have:

$$F_m^{(2)} = N_m(2) + N_m(1,1),$$
$$F_m^{(3)} = 2N_m(3) + 4N_m(2,1) + 2N_m(1^3).$$

Generators of the modules $N_m(3)$ are x_1^3 and $x_1x_2^2$; the modules with diagrams [2,1] are generated by $f_1 = x_1x_2x_1 - x_2x_1x_1$, $f_2 = x_1(x_2x_1) - x_2x_1^2$, $f_3 = x_1^2x_2 - x_2x_1x_1$ and $f_4 = x_1(x_1x_2) - x_2x_1^2$; generators of $N_m(1^3)$ are $S_{21}(x_1,x_2,x_3) = \sum_{\sigma\in S_3}(-1)^\sigma x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}$ and $S_{12}(x_1,x_2,x_3) = \sum_{\sigma\in S_3}(-1)^\sigma x_{\sigma(1)}(x_{\sigma(2)}x_{\sigma(3)})$, where $(-1)^\sigma$ means the sign of the permutation σ.

So identities are needed, which "glue" the isomorphic modules, so that the sum in $F_m^{(3)}(\mathfrak{M})$ will be of non-isomorphic ones only.

For the modules $N_m(3)$ such an identity is

(1) $$\alpha_1x^3 + \beta_1xx^2 = 0, \quad \text{for } (\alpha_1,\beta_1) \neq (0,0),\ \alpha_1,\beta_1 \in K.$$

It means that

(1^a) $$xx^2 = 0 \quad \text{if } \alpha_1 = 0, \text{ or}$$

(1^b) $$x^3 - kxx^2 = 0, \quad \text{where } k = \beta_1/\alpha_1 \text{ if } \alpha_1 \neq 0.$$

For $N_m(2,1)$ the following system has to be fulfilled:

(2) $$\begin{aligned} \gamma_{11}f_1 + \gamma_{12}f_2 + \gamma_{13}f_3 + \gamma_{14}f_4 &= 0\\ \gamma_{21}f_1 + \gamma_{22}f_2 + \gamma_{23}f_3 + \gamma_{24}f_4 &= 0\\ \gamma_{31}f_1 + \gamma_{32}f_2 + \gamma_{33}f_3 + \gamma_{34}f_4 &= 0\end{aligned}$$

and rank $A = 3$, where

$$A = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{14}\\ \gamma_{21} & \gamma_{22} & \gamma_{23} & \gamma_{24}\\ \gamma_{31} & \gamma_{32} & \gamma_{33} & \gamma_{34}\end{bmatrix}, \quad \gamma_{ij} \in K.$$

It means that the following identities hold:

(2^a) $f_i = 0$ for $i = 1,\dots,4$, or

(2^b) $f_i - k_if_4 = 0$ for $i = 1,2,3$ and for $k_1,k_2,k_3 \in K$, if $f_4 = 0$ is not an identity.

For the modules $N_m(1^3)$ the needed identity is

(3) $$\alpha_2S_{21}(x_1,x_2,x_3) + \beta_2S_{12}(x_1,x_2,x_3) = 0,$$

where $(\alpha, \beta_2) \neq (0,0)$, $\alpha_2, \beta_2 \in K$. It means that

(3^a) $S_{12}(x_1, x_2, x_3) = 0$ if $\alpha_2 = 0$, or

(3^b) $S_{21}(x_1, x_2, x_3) - pS_{12}(x_1, x_2, x_3) = 0$, where $p = \beta_2/\alpha_2$ if $\alpha_2 \neq 0$.

In the paper the following varieties are examined:

$\mathfrak{M}_1 = [*, k_1, k_2, k_3, *]$ with identities (1^a), (2^b) and (3^a),
$\mathfrak{M}_2 = [*, k_1, k_2, k_3, p]$ with identities (1^a), (2^b) and (3^b),
$\mathfrak{M}_3 = [k, k_1, k_2, k_3, *]$ with identities (1^b), (2^b) and (3^a),
$\mathfrak{M}_4 = [k, *, *, *, *]$ with identities (1^b), (2^a) and (3^a),
$\mathfrak{M}_5 = [k, *, *, *, p]$ with identities (1^b), (2^a) and (3^b),
$\mathfrak{M}_6$ — a variety, for which $P_3(\mathfrak{M}_6) = M(3)$.

For a multihomogeneous polynomial $f(x_1, \ldots, x_r)$ of degree λ_i in x_i we denote by

$$\operatorname{lin}(f) = f(x_1 \mathrm{I} y_{11}, \ldots, y_{1\lambda_1}; \ldots; x_r \mathrm{I} y_{r1}, \ldots, y_{r\lambda_r})$$

the linearization of $f(x_1, \ldots, x_m)$ which equals the multilinear in y_{ij} for $i = 1, \ldots, r$ component of

$$f(x_1 + y_{11} + \ldots + y_{1\lambda_1}, \ldots, x_r + y_{r1} + \ldots + y_{r\lambda_r}).$$

We point that $f = 0$ and $\operatorname{lin}(f) = 0$ are equivalent [6].

PROPOSITION 1.1. *Let M be an S_n-submodule of P_n and let Q be the set of the multilinear consequences of degree $n+1$ of the polynomial identities of M. Then Q is an S_{n+1}-module of P_{n+1} which is a homomorphic image of the S_{n+1}-module*

$$((M^{\downarrow} S_{n-1}) \otimes_K (M(2) + M(1^2))^{\uparrow} S_{n+1} + 2(M \otimes_K M(1))^{\uparrow} S_{n+1}, \text{ i.e.}$$

a) In the first summand S_{n-1} acts on the set $\{1, \ldots, n-1\}$ fixing n. S_2 acts on $\{n, n+1\}$, the tensor product is an $S_{n-1} \times S_2$-module, where the direct product $S_{n-1} \times S_2$ is canonically embedded in S_{n+1}.

b) The consequences $f(x_1, \ldots, x_n). x_{n+1}$ for $f \in M$ and $x_{n+1} \cdot f(x_1, \ldots, x_n)$ generate two factor-modules $M \otimes_K M(1)^{\uparrow} S_{n+1}$, where $S_n \times S_1$ is canonically embedded in S_{n+1}.

COROLLARY 1.2. *Let λ be a partition of n and let $M(\lambda) \subset P_n$. Then the S_{n+1}-module $M'(\lambda)$ of all multilinear consequences of $M(\lambda)$ in P_{n+1} equals $\sum \alpha_\mu M(\mu)$, where the non-negative integers λ_μ are bounded by the number of diagrams $[\mu]$ obtained by the following devices:*

a) We remove a box from $[\lambda]$ and obtain a diagram $[\nu]$. Then we add two new boxes to $[\nu]$ and produce a diagram $[\mu]$ such that these two new boxes do not belong to one and the same column of $[\mu]$ if we consider the module

$M(2)$ (or do not belong to one and the same row of $[\mu]$ if we consider the module $M(1^2)$),

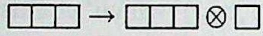
b) *We add a new box to $[\lambda]$ and obtain $[\mu]$.*

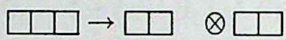
2. Consequences of degree 4 as linear combinations of the generators of the modules $N_m(\lambda)$

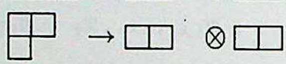
The symbols x, y, z, t will be used for the free generators of $K\{X\}$. The needed identities of degree 3 are written as:

$$\begin{aligned} &d_1 = x^3 - kxx^2 = 0 \\ &d_1 = xyx - yxx - k_1(x(xy) - yx^2) = 0 \\ (2.1)\qquad &d_3 = x(yx) - yx^2 - k_2(x(xy) - yx^2) = 0 \\ &d_4 = x^2y - yxx - k_3(x(xy) - yx^2) = 0 \\ &d_5 = S_{21}(x,y,z) - pS_{12}(x,y,z) = 0, \quad k, k_1, k_2, k_3, p \in K. \end{aligned}$$

$A : N_1(4)$. Due to 1.2 we can have the following diagrams:

(A1) 

(A2) 

(A3) 

For (A 1) we have $d_1x = 0$ and $xd_1 = 0$. For (A 2) we linearize partially d_1 and substitute $u = x^2$ i.e. $d_1(x\mathrm{I}x^2) = 0$. For (A 3) we consider $d_1(y = x^2)$ for $i = 2, 3, 4$. So we get the system:

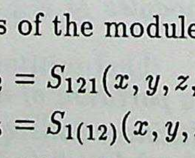
$$\begin{aligned} &x^4 - kxx^2x = 0 \\ &xx^3 - kx(xx^2) = 0 \\ (2.2)\qquad &x^4 + xx^2x + x^2x^2 - k(x^2x^2 + xx^3 + x(xx^2)) = 0 \\ &xx^2x - x^4 - k_1(x(xx^2) - x^2x^2) = 0 \\ &xx^3 - x^2x^2 - k_2(x(xx^2) - x^2x^2) = 0 \\ &x^2x^2 - x^4 - k_3(x(xx^2) - x^2x^2) = 0. \end{aligned}$$

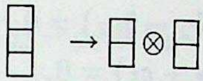
$B : N_4(1^4)$. The generators of the modules now are:

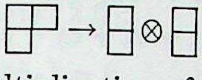
$$f_1 = S_{211}(x,y,z,t), \quad f_2 = S_{121}(x,y,z,t), \quad f_3 = S_{22}(x,y,z,t),$$
$$f_4 = S_{1(21)}(x,y,z,t), \; f_5 = S_{1(12)}(x,y,z,t).$$

The indices show the way of brackets in the standard polynomials, for example $S_{121} = \sum_{\sigma \in S_4} (-1)^\sigma x_{\sigma(1)}(x_{\sigma(2)}x_{\sigma(3)})x_{\sigma(4)}$.

According to Corollary 1.2 we have the following diagrams:

(B1) 

(B2) 

(B3) 

For (B 1) right and left multiplication of d_5 by t give

(b1) $$f_1 - pf_2 = 0,$$

(b2) $$f_4 - pf_5 = 0.$$

For (B 2) we consider $d_5(z = [z,t])$, in which three times a circling permutation of x, y, z, t is made alternating the signs. Then we transpose y and z in $d_5(z = [z,t])$ and again a circling of x, y, z, t is used. The sum of the six identities thus received leads to

(b3) $$f_1 - f_2 + f_3 - p(f_3 - f_4 + f_5) = 0.$$

For (B 3) in $d_2(x\mathrm{I}[z,t])$ we transpose x and y and then x and t. In the first circling permutation of x, y, z, t in $d_2(x\mathrm{I}[z,t])$ we transpose y and z and then x and y. In the second one the transpositions are of z and t and then of y and z. In the third one we transpose x and t and then z and t. So we come to 12 consequences, the sum of which is the identity

(b4) $$-f_1 + f_2 + 2f_3 - k_1(f_3 + 2f_4 + f_5) = 0.$$

Analogous transformations on d_3 and d_4 lead accordingly to:

(b5) $$-f_3 + f_4 + 2f_5 - k_2(f_3 + 2f_4 + f_5) = 0 \quad \text{and}$$

(b6) $$f_1 + 2f_2 + f_3 - k_3(f_3 + 2f_4 + f_5) = 0.$$

Identities (b1), ..., (b6) will be cited as (2.3) later on.

$C : N_2(3,1)$. The standard generators now are:

$a_1 = xyxx - yxxx$ and a_i $(1 = 2, \dots, 5)$ for brackets $(*(**)*)$, $(**)(**)$, $*(***)$ and $*(*(**))$;

$b_1 = x^2yx - yxxx$ and b_i $(i = 2, \dots, 5)$ for the respectvive brackets;

$c_1 = x^3y - yxxx$ and c_i $(i = 2, \dots, 5)$ for the respective brackets.

The system (2.5) in this case consists of identities

(c1) $2a_3 + b_1 - c_1 + c_2 - k_1(a_5 - b_3 + c_3 + 2c_4) = 0,$

(c2) $-a_1 + a_3 + b_1 + b_2 - k_1(-a_3 + c_3 + c_4 + c_5) = 0,$

(c3) $a_2 + a_3 + b_1 - k_1(b_5 + c_3 + c_4) = 0,$

(c4) $a_1 - k_1 b_2 = 0,$

(c5) $-a_4 + b_4 - k_1(-a_5 + c_5) = 0,$

(c6) $a_1 - a_2 + b_2 - b_3 + c_3 - k(a_3 - a_4 + b_4 - b_5 + c_5) = 0,$

(c7) $2a_1 - a_3 - b_1 + 2b_2 - c_1 - c_2 + 2c_3 - k(2a_3 - a_5 - b_3 + 2b_4 - c_3 - c_4 + 2c_5) = 0,$

(c8) $a_1 + b_1 - 3c_1 - k(a_2 + b_2 - 3c_2) = 0,$
(c9) $a_4 + b_4 + c_4 - k(a_5 + b_5 + c_5) = 0,$
(c10) $a_3 - b_1 + c_1 - c_2 - p(a_5 - b_3 + c_3 - c_4) = 0$

and those of the system (2.4):

$$
\begin{aligned}
&2a_5 + b_3 - c_3 + c_4 - k_2(a_5 - b_3 + c_3 + 2c_4) = 0\\
&-a_3 + a_5 + b_3 + b_4 - k_2(-a_3 + c_3 + c_4 + c_5) = 0\\
&a_4 + a_5 + b_3 - k_2(b_5 + c_3 + c_4) = 0\\
&a_2 - k_2 b_2 = 0\\
&-a_5 + b_5 - k_2(-a_5 + c_5) = 0\\
&a_3 - b_1 + c_1 + 2c_2 - k_3(a_5 - b_3 + c_3 + 2c_4) = 0\\
&-a_1 + c_1 + c_2 + c_3 - k_3(-a_3 + c_3 + c_4 + c_5) = 0\\
&b_3 + c_1 + c_2 - k_3(b_5 + c_3 + c_4) = 0\\
&b_1 - k_3 b_2 = 0\\
&-a_4 + c_4 - k_3(-a_5 + c_5) = 0.
\end{aligned}
\tag{2.4}
$$

$D : N_3(2,1^2)$. The standard generators now are:

$$f_1 = \sum_{\sigma\in S_3} (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} x_1, \qquad g_1 = \sum_{\sigma\in S_3} (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} x_1 x_{\sigma(3)},$$

$$h_1 = \sum_{\sigma\in S_3} (-1)^\sigma x_{\sigma(1)} x_1 x_{\sigma(2)} x_{\sigma(3)}.$$

For brackets $(*(**)*)$, $(**)(**)$, $*(***)$ and $*(*(**))$ the indices are $2,\ldots,5$, respectively.

The system for $N_3(2,1^2)$ is formed by identities

(d1) $-f_1 + f_3 + g_1 + g_2 - g_3 - h_2 - p(-f_3 + f_5 + g_3 + g_4 - g_5 - h_4) = 0,$
(d2) $f_1 - pf_2 = 0,$
(d3) $f_4 - g_4 + h_4 - p(f_5 - g_5 + h_5) = 0,$
(d4) $f_1 + f_3 - g_1 + g_2 + g_3 + 2h_1 + h_2 - p(f_3 + f_5 - g_3 + g_4 + g_5 + 2h_3 + h_4) = 0,$
(d5) $f_1 + 2f_2 + f_3 + g_1 - g_2 - g_3 + h_2 + 2h_3$
$\qquad -k(f_3 + 2f_4 + f_5 + g_3 - g_4 - g_5 + h_4 + 2h_5) = 0,$
(d6) $-f_1 + 3g_1 - k_1(f_2 + 3h_2) = 0,$
(d7) $2f_4 + g_4 - h_4 - k_1(f_5 + 2g_5 + h_5) = 0,$
(d8) $-f_1 + 2f_2 - g_1 - g_2 + h_2 - k_1(-f_3 + f_5 - g_3 - g_5 + 2h_5) = 0,$
(d9) $f_1 - 2f_3 - g_1 + g_2 - 2g_3 + 2h_1 + h_2$
$\qquad -k_1(-f_3 - f_5 + g_3 + 2g_4 - g_5 - 2h_3 + 2h_4) = 0,$

(d10) $f_1 - f_2 + f_3 - g_3 - k_1(g_3 - g_4 + h_4 - h_5) = 0$

and those of the following system (2.6):

$$
(2.6)\quad
\begin{aligned}
&-f_2 + 3g_2 - k_2(f_2 + 3h_2) = 0\\
&2f_5 + g_5 - h_5 - k_2(f_5 + 2g_5 + h_5) = 0\\
&-f_3 + 2f_4 - g_3 - g_4 + h_4 - k_2(-f_3 + f_5 - g_3 - g_5 + 2h_5) = 0\\
&f_3 - 2f_5 - g_3 + g_4 - 2g_5 + 2h_3 + h_4\\
&-k_2(-f_3 - f_5 + g_3 + 2g_4 - g_5 - 2h_3 + 2h_4) = 0\\
&f_3 - f_4 + f_5 - g_5 - k_2(g_3 - g_4 + h_4 - h_5) = 0\\
&f_1 + 3h_1 - k_3(f_2 + 3h_2) = 0\\
&f_4 + 2g_4 + h_4 - k_3(f_5 + 2g_5 + h_5) = 0\\
&-f_1 + f_3 - g_1 - g_3 + 2h_3 - k_3(-f_3 + f_5 - g_3 - g_5 + 2h_5) = 0\\
&-f_1 - f_3 + g_1 + 2g_2 - g_3 - 2h_1 + 2h_2\\
&-k_3(-f_3 - f_5 + g_3 + 2g_4 - g_5 - 2h_3 + 2h_4) = 0\\
&g_1 - g_2 + h_2 - h_3 - k_3(g_3 - g_4 + h_4 - h_5) = 0.
\end{aligned}
$$

Briefly the system of the consequences is denoted by (2.7).

$E : N_2(2^2)$. A standard generator in this case is

$$f_1 = \sum_{\sigma,\tau\in S_2} (-1)^{\sigma+\tau} x_{\sigma(1)}x_{\tau(1)}x_{\sigma(2)}x_{\tau(2)}.$$

For brackets $(*(**)*)$, $(**)(**)$, $*(***)$ and $*(*(**))$ the indices of the generators will be $2,\ldots,5$.

Another generator is

$$f_6 = \sum_{\sigma,\tau\in S_2} (-1)^{\sigma+\tau} x_{\sigma(1)}x_{\sigma(2)}x_{\tau(1)}x_{\tau(2)}$$

and for the corresponding way of brackets $f_7,\ldots,f_{10}$.

The system (2.9) in this case is formed by identities (e1), …, (e6) and those of the system (2.8):

(e1) $2f_1 - f_2 + 2f_3 - f_6 - f_8 - f_9 - k(2f_3 - f_4 + 2f_5 - f_7 - f_8 - f_{10}) = 0,$

(e2) $-f_2 + f_6 + f_8 + f_9 - p(-f_4 + f_7 + f_8 + f_{10}) = 0,$

(e3) $f_6 - k_1 f_2 = 0,$

(e4) $f_4 - f_7 - k_1 f_5 = 0,$

(e5) $f_2 - f_6 + 2f_8 - f_9 - k_1(2f_4 - 2f_7 + f_8 + f_{10}) = 0,$

(e6) $-2f_1 - f_2 + f_6 - f_9 - k_1(-2f_3 + 2f_5 + 2f_5 + f_8 - f_{10}) = 0,$

$$
(2.8)\quad
\begin{aligned}
&f_9 - k_2 f_2 = 0\\
&f_5 - f_{10} - k_2 f_5 = 0\\
&f_4 - f_7 - f_8 + 2f_{10} - k_2(2f_4 - 2f_7 + f_8 + f_{10}) = 0\\
&-2f_3 - f_4 - f_7 + f_8 - k_2(-2f_3 + 2f_5 + f_8 - f_{10}) = 0\\
&f_1 - k_3 f_2 = 0\\
&f_4 - k_3 f_5 = 0\\
&2f_2 + f_6 + f_8 - 2f_9 - k_3(2f_4 - 2f_7 + f_8 + f_{10}) = 0\\
&-2f_1 + 2f_3 + f_6 - f_8 - k_3(-2f_3 + 2f_5 + f_8 - f_{10}) = 0.
\end{aligned}
$$

3. Description of $P_4(\mathfrak{M}_i)$ for $i = 1, \ldots, 5$ and $P_n(\mathfrak{M}_6)$

Having already obtained the homogeneous linear systems for the standard generators of every module $N_m(\lambda)$, we determine the rank of the matrix of the corresponding system in any of the considered cases. If the system has a trivial solution only, there is no module with the corresponding diagram in $P_4(\mathfrak{M}_i)$. If the rank is not maximal, we define the multiplicities $k(\lambda)$ in the decomposition of $P_4(\mathfrak{M}_i)$ into a sum of irreducible modules i.e. in

$$
(3.1)\quad
\begin{aligned}
P_4(\mathfrak{M}_i) = &\, k(4)M(4) + k(1^4)M(1^4) + k(3,1)M(3,1)\\
&+ k(2,1^2)M(2,1^2) + k(2^2)M(2^2)
\end{aligned}
$$

THEOREM 3.1. *For $\mathfrak{M}_1 = [*, k_1, k_2, k_3, *]$ in (3.1)*

a) $k(4) \leq 1$ *if,* $[*, k_1, 2, k_1 - 1, *]$ *and a generator of the module $M(4)$ is the complete linearization of x^2x^2,*

b) $k(1^4) \leq 1$ *if* $[*, k_1, 0, 1 - k_1, *]$ *and the module $M(1^4)$ is generated by* $S_{1(21)}(x_1, x_2, x_3, x_4)$,

c) $k(3,1) = 0$,

d) $k(2,1^2) \leq 1$ *if* $[*, 0, 1, -1, *]$, *where a generator of $M(2,1^2)$ is the linearization of* $\sum_{\sigma \in S_3} (-1)^\sigma x_{\sigma(1)}(x_{\sigma(2)}(x_{\sigma(3)}x_1))$,

e) $k(2^2) = 0$.

f) *Otherwise the variety $\mathfrak{M}_1$ is nilpotent of index 4.*

Proof. We consider the corresponding GL_m-modules.

(4): The system (2.2) leads to the following matrix of the coefficients of the generators x^2x^2, xx^3 and x^4:

$$
A = \begin{bmatrix} 1 & 1 & 0\\ k_1 & 0 & -1\\ k_2 - 1 & 1 & 0\\ k_3 + 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} k_1 & 0 & -1\\ 1 & 1 & 0\\ k_2 - 2 & 0 & 0\\ k_3 - k_1 + 1 & 0 & 0 \end{bmatrix}.
$$

Rank $A = 2$ gives the conditions on k_2 and k_3 in a).

(1^4): In this case the first three identities of (2.3) are changed, namely $f_2 = 0, f_5 = 0, f_3 - f_4 = 0$ and the matrix of the coefficients of f_1, f_3 and f_4 is the following:

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 2-k_1 & -2k_1 \\ 0 & -1-k_2 & 1-2k_2 \\ 1 & 1-k_3 & -2k_3 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ -1 & 2-k_1 & 2-3k_1 \\ 0 & -1-k_2 & -3k_2 \\ 0 & 3-k_1-k_3 & 3-3k_1-3k_3 \end{bmatrix}$$

(3.1): Corresponding to $(c6), \ldots, (c10)$ of (2.5) are now the identities:

$$a_5 + b_5 + c_5 = 0$$
$$a_5 - b_3 + c_3 - c_4 = 0$$
$$a_3 - a_4 + b_4 - b_5 + c_5 = 0$$
$$2a_3 - a_5 - b_3 + 2b_4 - c_3 - c_4 + 2c_5 = 0$$
$$a_3 + b_2 - 3c_2 = 0$$

The system has a trivial solution only.
$(2, 1^2)$: Corresponding to $(d1), \ldots, (d5)$ of (2.7) are:

$$-f_3 + f_5 + g_3 + g_4 - g_5 - h_4 = 0$$
$$f_2 = 0$$
$$f_5 - g_5 + h_5 = 0$$
$$f_3 + f_5 - g_3 + g_4 + g_5 + 2h_3 + h_4 = 0$$
$$f_3 + 2f_4 + f_5 + g_3 - g_4 - g_5 + h_4 + 2h_5 = 0.$$

Easily we get the following matrix $A_{17\times 12}$:

$$\begin{bmatrix}
0 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & -1 \\
0 & 1 & 0 & 2 & 0 & -1 & 1 & 0 & 0 & 2 & 1 & 1 \\
0 & 1 & 2 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & 1 \\
-1 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & -3k_1 & 0 & 0 & 0 \\
0 & 0 & -2 & 3k_1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 3k_1 \\
-1 & k_1 & 0 & 0 & -1 & k_1 & 0 & 0 & 1-k_2 & 0 & 0 & -k_1 \\
1 & k_1-2 & 0 & 2k_1 & -1 & -2-k_1 & -2k_1 & 2 & 1+k_2 & 2k_1 & -2k_1 & k_1 \\
1 & 1 & 0 & 0 & 0 & -1-k_1 & k_1 & 0 & 0 & 0 & -k_1 & k_1 \\
0 & 0 & 0 & k_2-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & k_2 \\
0 & k_2-1 & 2 & 0 & 0 & k_2-1 & -1 & 0 & 0 & 0 & 1 & -k_2 \\
0 & 1+k_2 & 0 & 2k_2-4 & 0 & -1-k_2 & 1-2k_2 & 0 & 0 & 2+2k_2 & 1-2k_2 & k_2-2 \\
0 & 1 & -1 & 0 & 0 & -k_2 & k_2 & 0 & 0 & 0 & -k_2 & k_2-1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -3k_3 & 0 & 0 & 0 \\
0 & 0 & 1 & -3k_3 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & -3k_3 \\
-1 & 1+k_3 & 0 & 0 & -1 & k_3-1 & 0 & 0 & 0 & 2 & 0 & -k_3 \\
-1 & k_3-1 & 0 & 2k_3 & 1 & -1-k_3 & -2k_3 & -2 & 2+2k_2 & 2k_3 & -2k_3 & k_3 \\
0 & 0 & 0 & 0 & 1 & -k_3 & k_3 & 0 & 1-k_2 & -1 & -k_3 & k_3
\end{bmatrix}$$

Investigations on its rank give the result.

(2^2). Now the first two identities in (2.9) are

$$2f_3 - f_4 + 2f_5 - f_7 - f_8 - f_{10} = 0 \quad \text{and} \quad -f_4 + f_7 + f_8 + f_{10} = 0$$

The system has a trivial solution only.

Proof of condition f). The partial linearization of $x^4 = 0$ and the identities $a_1 = b_1 = c_1 = 0$ lead to $yxxx = xyxx = x^2yx = x^3y = 0$. New partial linearizations in x and $f_1 = f_6 = 0$ (generators for $N_2(2^2)$) lead to

$$x^2yy = xyyx = yxxy = y^2xx = xyxy = yxyx = 0.$$

In a similar way the partial linearizations of these identities in y and $f_1 = g_1 = h_1 = 0$ (generators of $N_3(2,1^2)$) lead to $zxxy = xyxz = x^2yz = xyzx = yxzx = yzxx = 0$.

New linearizations and $f_1 = 0$ (a generator of $N_4(1^4)$) give $zytx = 0$. Another way of brackets is treated anagously.

THEOREM 3.2. *For $\mathfrak{M}_2 = [*, k_1, k_2, k_3, p]$ the conditions on the multiplicities of the modules in (3.1) are the following:*

a) $k(4) \le 1$ *if* $[*, k_1, 2, k_1 - 1, p]$. *The module $M(4)$ is generated by the complete linearization of x^2x^2,*

b) $k(1^4) \le 1$ *if* $[*, k_1, 2 - k_1, k_1, 0]$, $[*, k_1, 1 - k_1, k_3, 1]$ *or*

$$[*, k_1, k_2, k_3, p \ne 0, 1 : k_2(p^2 - 2p - 1) = pk_1 + k_1 - 2,$$
$$k_1(p^2 + 3p + 1) = 2 + p - k_3 - pk_3].$$

The module $M(1^4)$ is generated by $S_{1(12)}(x_1, x_2, x_3, x_4)$,

c) $k(3,1) = 0$,

d) $1 \le k(2,1^2) \le 2$ *if* $[*, -2, -1, -1, -1]$. *Module generators are the linearizations of $\sum_{\sigma \in S_3} (-1)^\sigma (x_{\sigma(1)} x_{\sigma(2)})(x_1 x_{\sigma(3)})$ and $\sum_{\sigma \in S_3} (-1)^\sigma x_{\sigma(1)} \times (x_{\sigma(2)} x_1 x_{\sigma(3)})$,*

e) $k(2^2) \le 1$ *if* $[*, 0, 1, -1, p]$. *$M(2^2)$ is generated by the complete linealization of $\sum_{\sigma,\tau \in S_2} (-1)^{\sigma+\tau} x_{\sigma(1)}(x_{\tau(1)}(x_{\sigma(2)} x_{\tau(2)}))$. The same unequality holds for both the cases $[*, 2, -1, 1, 1]$ and $[*, -2, -1, -1, 5]$ and the module $M(2^2)$ is generated by the complete linearization of $\sum_{\sigma,\tau \in S_2} (-1)^{\sigma+\tau} x_{\sigma(1)} x_{\sigma(2)} \times (x_{\tau(1)} x_{\tau(2)})$.*

f) *Otherwise $\mathfrak{M}_2$ is a nilpotent variety of index 4.*

THEOREM 3.3. *For $\mathfrak{M}_3 = [k, k_1, k_2, k_3, *]$ we have in (3.1):*

a) $k(4) \le 1$ *if* $[0, k_1, k_1, -k_1, *]$, $[1, k_1, k_2 \ne 1, k_3, *]$ *or*

$$[k \ne 0; 1, k_1, k_2, k_3, * : k_3(k-1) = k_2 - k + kk_1,\ 2k_3 = 2k + k_1 - (3+k)k_2],$$

$$k(4) = 1 \quad \text{if } [1, k_1 \ne 0, k_2, k_3, *] \text{ or } [1, k_1, k_2, k_3 \ne -1, *].$$

$M(4)$ is generated by the complete linearization of

$x(xx^2)$, $1 \leq k(4) \leq 2$ *if* $[1,0,1,-1,*]$. *The two generators are complete linearizations of* xx^2x *and* $x(xx^2)$,

b) $k(1^4) \leq 1$ *if* $[k, k_1, 0, 1-k_1, *]$. *The module* $M(1^4)$ *is generated by* $S_{1(12)}(x_1, x_2, x_3, x_4)$,

c) $k(3,1) \leq 1$ *if* $[0,0,0,0,*]$ *or* $[1,0,0,1,*]$. *In the first case* $M(3,1)$ *is generated by the complete linearization of* $\sum_{\sigma \in S_2}(-1)^\sigma x_{\sigma(1)}(x_1(x_1x_{\sigma(2)}))$ *or by that of* $\sum_{\sigma\in S_2}(-1)^\sigma x_{\sigma(1)}x_1x_1x_{\sigma(2)}$ *in the second one,*

d) $k(2,1^2) = 0$,

e) $k(2^2) \leq 1$ *if* $[k \neq 1, 0, 1, -1, *]$. *The module* $M(2^2)$ *is generated by the complete linearization of* $\sum_{\sigma,\tau\in S_2}(-1)^{\sigma+\tau}(x_{\sigma(1)}x_{\tau(1)})(x_{\sigma(2)}x_{\tau(2)})$,

$k(2^2) = 2$ *if* $[1,0,1,-1,*]$. *Generators are the complete linearizations of* $\sum_{\sigma,\tau\in S_2}(-1)^{\sigma+\tau}(x_{\sigma(1)}x_{\tau(1)})(x_{\sigma(2)}x_{\tau(2)})$ *and of* $\sum_{\sigma,\tau\in S_2}(-1)^{\sigma+\tau}x_{\sigma(1)} \times (x_{\tau(1)}x_{\sigma(2)}x_{\tau(2)})$.

f) *Otherwise the variety* $\mathfrak{M}_3$ *is nilpotent of index 4.*

THEOREM 3.4. *If in* $\mathfrak{M}_4$ *and* $\mathfrak{M}_5$ $k \neq 1$ *then* $P_4(\mathfrak{M}_4) = \{0\}$ *and* $P_4(\mathfrak{M}_5) = \{0\}$. *For* $k = 1$ $P_4(\mathfrak{M}_4) = M(4)$ *and* $P_4(\mathfrak{M}_5) = M(4)$.

THEOREM 3.5. $P_n(\mathfrak{M}_6) = M(n)$ *for* $n \geq 3$.

Proof. Because of $f_i=0$ $(i=1,\ldots,4)$ and $S_{12}=S_{21}=0$ $x_1\ldots x_kx_{k+1}\ldots x_4 - x_1\ldots x_{k+1}x_k\ldots x_4 \in P_4(\mathfrak{M}_6)\cap T(\mathfrak{M}_6)$ for any brackets save $(**)(**)$. In the last case we refer to the corresponding to (2.2), (2.3), (2.5), (2.7) and (2.9) systems and get $P_4(\mathfrak{M}_6) = M(4)$. Using induction on n we see that

$$x_1\ldots(x_kx_{k+1})\ldots x_n - x_1\ldots(x_{k+1}x_k)\ldots x_n \in P_n \cap T(\mathfrak{M}_6).$$

If $k > 1$ then

$$\begin{aligned}
&(x_1(\ldots x_k))(x_{k+1}x_{k+2}\ldots x_n) - (x_1(\ldots x_{k+1}))(x_kx_{k+2}\ldots x_n)\\
&= x_{k+1}(x_1(\ldots x_k)x_{k+2}\ldots x_n) - x_k(x_1(\ldots x_{k+1})x_{k+2}\ldots x_n)\\
&= x_{k+1}(x_1(\ldots x_{k+2})x_k\ldots x_n) - x_k(x_1(\ldots x_{k+2})x_{k+1}\ldots x_n)\\
&= (x_1(\ldots x_{k+2}))(x_{k+1}x_k\ldots x_n) - (x_1(\ldots x_{k+2}))(x_kx_{k+1}\ldots x_n) = 0.
\end{aligned}$$

For $k = 1$

$$\begin{aligned}
&x_1(x_2, x_3\ldots x_n) - x_2(x_1x_3\ldots x_n)\\
&= (x_2x_3)(x_1x_4\ldots x_n) - (x_1x_3)(x_2x_4\ldots x_n)\\
&= (x_2(x_1x_4))(x_3x_5\ldots x_n) - (x_1(x_2x_4))(x_3x_5\ldots x_n) = 0.
\end{aligned}$$

All the equalities are modulo $P_{n-1}\cap T(\mathfrak{M}_6)$. So $P_n(\mathfrak{M}_6) = M(n)$.

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ENDOMORPHISM CONGRUENCES

Introduction

Considered in this paper the notion of endomorphism congruence has a close connection with a notion of result. To say most generally, results are fractions such that their numerators are elements of a certain linear space while their denominators are injective endomorphisms of that space. Generally, division by an endomorphism is lead out of range of elements. In connection with it we obtain the regular and singular results. Using the endomorphism congruence properties we can decide about regularity of certain results on the basis of the others confirming, if they are regular or singular.

The adequate examples in the Bittner operational calculus are given in this article.

1. Results and operators [1], [2]

Let $L(X,X)$ be the space of endomorphisms of a linear space X (over a field Γ). Moreover, let $\pi(X)$ be a multiplicative and commutative semigroup of injective endomorphisms of $L(X,X)$.

When X and $\pi(X)$ are given, we can introduce ordered pairs

$$\xi := [x, A],\ x \in X,\ A \in \pi(X)$$

and the quality relation

$$([x,A] = [y,B]) \overset{\text{def}}{\Longleftrightarrow} (Bx = Ay),\ x, y \in X,\ A, B \in \pi(X)$$

which is of equivalence type.

By this relation the entire set of considered pairs is divided into equivalence classes. These classes are called results. An arbitrary representative $\xi = [x, A]$ of such a class is also called a result. For such representative the fraction symbol $\xi = \frac{x}{A}$ is applied.

If in the set of results $\Xi(X,\pi(X))$ (denoted briefly by $\Xi(X)$) we introduce the operations

$$\frac{x}{A}+\frac{y}{B} := \frac{Bx+Ay}{AB}, \quad \gamma\left(\frac{x}{A}\right) := \frac{\gamma x}{A},$$

where $x,y \in X$, $\gamma \in \Gamma$, $A,B \in \pi(X)$, then $\Xi(X)$ is a linear space over the field Γ. The elements of X can be identified with the results, since the map

$$x \mapsto \frac{Ax}{A},\ x \in X,\ A \in \pi(X)$$

is an isomorphism.

The elements $x \in X$ are called regular results and the elements $\xi \in \Xi(X) \setminus X$ are called singular results (cf [6]).

In $\Xi(X)$ we can also define the operation

$$B\left(\frac{x}{A}\right) := \frac{Bx}{A},$$

where $x \in X$, $A \in \pi(X)$, $B \in L(X,X)$, $AB = BA$.

With the given endomorphism $R \in L(X,X)$ commutative with the operations $A \in \pi(X)$, the linear operation given by the formula

$$\mu\frac{x}{B} := \frac{Rx}{AB}$$

is called an operator on the results space $\Xi(X)$ and denoted as $\mu = \frac{R}{A}$. The operator $\mu_0 = \frac{AR}{A}$ is identified with the endomorphism R.

The operator sum, the product of an operator by an element of Γ and the superposition of operators are operators. The division by an operator, the numerator of which is an injection, defined as the product by the inverse of the operator, is also an operator.

2. Endomorphism congruences

Let $End(X)$ be a commutative algebra of endomorphisms of $L(X,X)$ (with the usual multiplication of endomorphisms).

It is said that two endomorphisms $A,B \in End(X)$ are congruent by the modulus $M \in \pi(X)$ when there exists an endomorphism $K \in End(X)$ such that

$$(A-B)x = KMx$$

for all $x \in X$. Then we denote

$$A \equiv B \pmod{M}.$$

This relation is a congruence. If $A \equiv B \pmod{M}$, then the elements

$$\frac{(A-B)x}{M}$$

are the regular results for all $x \in X$, i.e. the operator $\frac{A-B}{M}$ is an endomorphism of X.

The set

$$J_M := \{KM : K \in End(X)\}, \ M \in \pi(X)$$

is an ideal of the ring $End(X)$.

When $A, B \in End(X)$, we can write

$$A \equiv B \ (\operatorname{mod} J_M),$$

if $A - B \in J_M$. Therefore

$$A \equiv B \ (\operatorname{mod} M) \leftrightarrow A \equiv B \ (\operatorname{mod} J_M).$$

EXAMPLE 1. Let X be a certain real linear space. Moreover, let $End(X)$ be an algebra of endomorphisms

$$Ax := a \cdot x, \quad x \in X,$$

where a is a given integer. The modulus $M \in \pi(X)$ is defined by the formula

$$Mx := m \cdot x, \quad x \in X,$$

where m is a given natural number.

Then

$$A \equiv B \ (\operatorname{mod} M)$$

if and only if

$$a \equiv b \ (\operatorname{mod} m)$$

in the classical congruence of integers sense.

EXAMPLE 2. Let $X := C^0(Q, R^1)$, where $Q \subset R^1$. The set $End(X)$ is defined as the algebra of endomorphisms

$$Ax := \{A(t)x(t)\}, \quad x = \{x(t)\} \in X,$$

where $\{A(t)\} \in X$ is a given polynomial.

The modulus $M \in \pi(X)$ is defined as the endomorphism

$$Mx := \{M(t)x(t)\}, \quad x = \{x(t)\} \in X,$$

where the given polynomial $\{M(t)\} \in X$ satisfies the condition $M(t) \neq 0$ for all $t \in Q$.

Then

$$A \equiv B \ (\operatorname{mod} M)$$

if and only if $\{M(t)\}$ is a divisor of the polynomial $\{A(t) - B(t)\}$.

Obviously the same modulus congruences can be added, substracted and multiplied by sides. It can be generalized by induction for any finite number

of congruences. In particular, the both sides of a congruence can be multiplied by the same endomorphism and they can be also raised to the same power with a natural exponent.

Therefore, if $A_i, U \in End(X)$, $i = 0, 1, \ldots, n$ and

$$W(U) := A_0 + A_1 U + \ldots + A_n U^n,$$

then $W(U) \in End(X)$ and

(1) $$A \equiv B \pmod{M} \quad \text{implies} \quad W(A) \equiv W(B) \pmod{M}.$$

3. Operational calculus

In accordance with notation used e.g. in [2], the Bittner operational calculus is the system

$$CO(L^0, L^1, S, T_q, s_q, q, Q),$$

where L^0 and L^1 are linear spaces over a field Γ.

The linear operation $S : L^1 \to L^0$ (denoted as $S \in L(L^1, L^0)$), called the (abstract) derivative, is a surjection. Moreover, Q is a nonempty arbitrary set of indices q for the operations $T_q \in L(L^0, L^1)$ such that $ST_q f = f$, $f \in L^0$, called integrals, and for the operations $s_q \in L(L^1, L^1)$ such that $s_q x = x - T_q Sx$, $x \in L^1$, called limit conditions. The kernel of S, i.e. the set $Ker\, S := \{c \in L^1 : Sc = 0\}$, is called the space of constants for the derivative S.

Assume that $L^1 \subset L^0$. Then

$$Ker\, S \subset L^1 \subset L^0$$

and the integrals T_q, $q \in Q$ are endomorphisms of L^0. The iterations of these operations can be also formed.

4. Examples of congruences in the operational calculus

A. It is not difficult to check that any integral T_q, $q \in Q$ is an injection of L^0. None of two endomorphisms $A, B, A \neq B$ of L^0 and $Ker S$ are congruent by the modulus T_q, because

$$\xi = \frac{(A - B)c}{T_q}$$

is a singular result for all $c \in Ker\, S \setminus \{0\}$. In reality, if $d := (A - B)c$, where $c \in Ker\, S \setminus \{0\}$, then $d \in Ker\, S \setminus \{0\}$ and $T_q \xi = d$. Then with $\xi \in L^0$ it would be $s_q T_q \xi = s_q d = d$ and hence $d = 0$, what is impossible (cf [2]).

B. If R is an endomorphism of L^0 and L^1, commutative with the derivative S and the limit condition s_q, then R is commutative with the integral T_q.

Moreover, if an abstract differential equation

(2) $$Sx = Rx, \quad x \in L^1$$

with the limit condition

$$s_q x = 0$$

has only zero solution, then R is called the q-logarithm.

If R is a q-logarithm, then there exists a semigroup $\pi(L^0)$ such that $I - T_q R \in \pi(L^0)$, where $I := id_{L^0}$ and the result

$$\xi = \frac{c}{I - T_q R}, \quad c \in Ker\, S$$

is well defined.

If the result ξ is regular, then it is called an exponential element and denoted by the symbol $e^{Rt_q} c$ (see [2], cf [3, 6]).

The exponential element $x = e^{Rt_q} c$ is a solution of the equation (2) with the limit condition $s_q x = c$. Moreover, for every $n \in N$ we have

$$x = c + T_q Rc + \ldots + T_q^n R^n c + T_q^{n+1} R^{n+1} x.$$

The expression

$$w_n = c + T_q Rc + \ldots + T_q^n R^n c$$

is called the n-th Taylor polynomial for the exponential element $x = e^{Rt_q} c$ (in the point $q \in Q$).

Let

$$W(U) := I + U + \ldots + U^n,$$

where $n \in N$, $I := \mathrm{id}_{L^0}$, $U \in End(L^0)$.

Then

$$w_n = W(T_q R)c.$$

Since

$$I \equiv T_q R \ (\mathrm{mod}\,(I - T_q R)),$$

so, on the basis of (1), the result

$$\eta = \frac{w_n}{I - T_q R}$$

is regular if and only if ξ is the regular result.

C. Assume that the abstract differential equation

$$S^2 x + Sx + x = 0, \quad x \in L^2 := \{x \in L^1 : Sx \in L^1\}$$

with the limit conditions

$$s_q x = 0, \quad s_q Sx = 0$$

has only zero solution.

Then there exists a semigroup $\pi(L^0)$ such that $I + T_q + T_q^2 \in \pi(L^0)$. By induction it can be proved that the congruence

$$(I + T_q)^{2n+1} \equiv -T_q^{n+2} \ (\mathrm{mod}\,(I + T_q + T_q^2))$$

holds for all non-negative integer n.

Due to the above mentioned the solution of the abstract integral equation

$$(I + T_q + T_q^2)\xi = (I + T_q)^{2n+1} f, \quad \xi \in \Xi(L^0),$$

where $f \in L^0$ is a given element and n is an arbitrary but fixed non-negative integer, is a regular result if and only if the solution of the equation

$$(I + T_q + T_q^2)\eta = -T_q^{n+2} f, \quad \eta \in \Xi(L^0)$$

is a regular result.

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PRODUCT FINAL DIFFERENTIAL STRUCTURES ON THE PLANE AND PRINCIPAL-DIRECTED CURVES

Product final differential structures $\mathcal{S}^1$ and $\mathcal{S}^2$ on $\mathbb{R}^2$ were defined in paper [2]. The differential spaces $\mathbb{R}\times_1\mathbb{R} = (\mathbb{R}^2, \mathcal{S}^1)$ and $\mathbb{R}\times_2\mathbb{R} = (\mathbb{R}^2, \mathcal{S}^2)$ have many common properties and they can be considered together as the differential space $\mathbb{R}\times_k\mathbb{R} = (\mathbb{R}^2, \mathcal{S}^k)$ where $k = 1$ or $k = 2$. In the above-mentioned paper it was proved that every regular curve in $\mathbb{R}\times_k\mathbb{R}$ is contained in a principal line, i.e. a straight line which is vertical or horizontal. This leads to a characterization of such curves as regular ones in $\mathbb{R}^2$ which are contained in principal lines. It is easily seen that every smooth curve in $\mathbb{R}\times_k\mathbb{R}$ is smooth in $\mathbb{R}^2$, but not conversely in general. In this paper we present a characterization of arbitrary smooth curves in $\mathbb{R}\times_k\mathbb{R}$ as some smooth ones in $\mathbb{R}^2$ (Theorem 2.20). It turns out that the characterization obtained does not depend on k (Corollary 2.21).

In Section 1 we first observe that every smooth curve in $\mathbb{R}\times_k\mathbb{R}$ is principal-directed in $\mathbb{R}^2$, however, there are smooth curves in $\mathbb{R}^2$ which are not smooth in $\mathbb{R}\times_k\mathbb{R}$ (Example 2.22). For this reason, we start with considerations of principal-directed curves in $\mathbb{R}^2$. Next, we distinguish and study certain classes of such curves, especially, the class of locally K-subordinate curves which is exactly that of all smooth ones in $\mathbb{R}\times_k\mathbb{R}$ (Theorem 2.20).

In Section 2 we introduce in different ways the classes of locally K-subordinate sets in $\mathbb{R}^2$ and of C^∞ subsets of $\mathbb{R}\times_k\mathbb{R}$. We prove that these classes are identical (Theorem 2.15) and show that they can be used for a characterization of smooth curves in $\mathbb{R}\times_k\mathbb{R}$. Moreover, it turns out that such curves are proper for a characterization of smooth maps from $\mathbb{R}\times_k\mathbb{R}$ to $\mathbb{R}\times_\ell\mathbb{R}$ where $k,\ell\in\{1,2\}$ (Propositions 2.24 and 2.25). By definitions, the class of principal-directed curves (locally K- subordinate sets) in $\mathbb{R}^2$ and its subclasses considered here do not depend on $\mathbb{R}\times_k\mathbb{R}$. Since the major part of this paper is devoted to the study of such classes, therefore this

portion of our paper has respect to the classical differential geometry on the plane.

Clearly, one may generalize considerations from this paper to those for $\mathbb{R}^n$ where $n > 2$ (compare [2], Section 5). It seems that such generalizations can, to a considerable extent, be obtained as the corresponding combinatorial n-variants with respect to our case $n = 2$. However, we must be careful whether direct generalized properties can hold since the topological and differential structures of $\mathbb{R}^n$ $(n > 2)$ are much more complicated than the corresponding ones of the plane.

1. Locally K-subordinate curves

In what follows, $k = 1, 2$ is fixed but arbitrary. First, we recall the definition of the differential structure $\mathcal{S}^k$ on $\mathbb{R}^2$ (see [2]). For any $a, b \in \mathbb{R}$ consider the maps $i_b : \mathbb{R} \to \mathbb{R}^2$ and $j_a : \mathbb{R} \to \mathbb{R}^2$ defined by $i_b(x) = (x, b)$ and $j_a(y) = (a, y)$. Let us denote by $\mathcal{F}^k$ the family of all real functions (real continuous functions) on $\mathbb{R}^2$ when $k = 1$ $(k = 2)$. We set

$$\mathcal{S}^k = \{\alpha \in \mathcal{F}^k : \alpha \circ i_b \in C^\infty(\mathbb{R}) \wedge \alpha \circ j_a \in C^\infty(\mathbb{R})\ \forall a, b \in \mathbb{R}\}.$$

It is seen that $\mathcal{S}^k$ is a differential structure on $\mathbb{R}^2$ and the differential space $\mathbb{R} \times_k \mathbb{R}$ is defined to be the pair $(\mathbb{R}^2, \mathcal{S}^k)$. We shall regard $\mathbb{R} \times_k \mathbb{R}$ as a topological space under the Sikorski topology defined to be the weakest one on $\mathbb{R}^2$ in which all functions from $\mathcal{S}^k$ are continuous (see [3], §14).

By an interval of $\mathbb{R}$ we will mean a nonsingle (i.e. nonsingle-element) connected subspace of $\mathbb{R}$. Every curve in $\mathbb{R}^2$ is assumed to be a continuous map $c : I \to \mathbb{R}^2$ where $I = \mathrm{dom}(c)$ is an interval of $\mathbb{R}$. A map $c : I \to \mathbb{R} \times_k \mathbb{R}$ is called a *smooth curve* in $\mathbb{R} \times_k \mathbb{R}$ if it is a smooth map of differential spaces where $I = \mathrm{dom}(c)$ is an interval of $\mathbb{R}$ regarded as a differential space under the natural structure induced from $\mathbb{R}$. Since $\mathcal{S}^k$ contains all real smooth functions on $\mathbb{R}^2$, it follows that every smooth curve in $\mathbb{R} \times_k \mathbb{R}$ is smooth in $\mathbb{R}^2$ (in the usual sense).

Let $c = (\alpha, \beta) : I \to \mathbb{R}^2$ be a smooth curve, that is, $\alpha, \beta \in C^\infty(I)$, we define the k-th derivative of c at $s \in I$ to be the vector $(D^k c)(s) = [D^k(\alpha)(s), D^k(\beta)(s)]$. By $\dot{c} = [\dot{\alpha}, \dot{\beta}] = D^1 c$ will be denoted the *canonical vector field tangent* to c. We call c *regular* (*stationary*) at s if $\|\dot{c}(s)\| > 0$ $(\|\dot{c}(s)\| = 0)$, where $\|\dot{c}(s)\| = (\dot{\alpha}(s)^2 + \dot{\beta}(s)^2)^{1/2}$. Denote by $\mathrm{dom}_R(c)$ $(\mathrm{dom}_S(c))$ the set of all regular (stationary) parameters of c. If $\mathrm{dom}_R(c) = \mathrm{dom}(c)$ $(\mathrm{dom}_S(c) = \mathrm{dom}(c))$, then c is called *regular* (*totally stationary*). We say that c is *completely stationary* at s or that s is a *singular parameter* of c if $(D^k c)(s) = [0, 0]$ for all $k \geq 1$. The set of all such parameters of c will be denoted by $\mathrm{dom}_{CS}(c)$. Obviously, $\mathrm{dom}_R(c)$ is an open subset of $\mathrm{dom}(c)$ but $\mathrm{dom}_S(c)$ and $\mathrm{dom}_{CS}(c)$ are closed subsets of $\mathrm{dom}(c)$. We call c

V-directed (*H-directed*) at s in case the vector $\dot{c}(s)$ is vertical (horizontal). Moreover, c is called *P-directed* at s if it is V-directed or H-directed at s. If $X \in \{V, H, P\}$, we denote by $\mathrm{dom}_X(c)$ the set of all parameters s of c such that c is X-directed at s. It is seen that $\mathrm{dom}_X(c)$ is a closed subset of $\mathrm{dom}(c)$. We say that c is *X-directed* in case $\mathrm{dom}_X(c) = \mathrm{dom}(c)$. Moreover, a P-directed curve will also be called *principal-directed.* These definitions immediately imply

1.1. LEMMA. *If c is a smooth curve in $\mathbb{R}^2$, then the following equalities hold*:

(a) $\mathrm{dom}_V(c) \cup \mathrm{dom}_H(c) = \mathrm{dom}_P(c)$;
(b) $\mathrm{dom}_V(c) \cap \mathrm{dom}_H(c) = \mathrm{dom}_S(c)$. ■

Clearly, this lemma implies

1.2. COROLLARY. *Every regular P-directed curve in $\mathbb{R}^2$ is V-directed or H-directed.* ■

If $X \in \{V, H\}$, then by an *X-principal line* we shall mean a straight line in $\mathbb{R}^2$ which is vertical if $X = V$ and horizontal if $X = H$. In turn, by a *(P-)principal line* we shall mean a straight line in $\mathbb{R}^2$ which is vertical or horizontal.

Let $X \in \{V, H, P\}$. A curve c in $\mathbb{R}^2$ is called *locally X-subordinate* at a parameter s if there are a neighbourhood U of s in $\mathrm{dom}(c)$ and an X-principal line L such that $c(U) \subseteq L$. The set of all such parameters of c will be denoted by $\mathrm{loc}_X(c)$. Obviously, $\mathrm{loc}_X(c)$ is an open subset of $\mathrm{dom}(c)$. Let the symbol int stand for the interior operation in $\mathrm{dom}(c)$. By an easy verification we get

1.3. LEMMA. *If c is a smooth curve in $\mathbb{R}^2$, then the following conditions hold*:

(a) $\mathrm{loc}_V(c) = \mathrm{int}\,\mathrm{dom}_V(c)$;
(b) $\mathrm{loc}_H(c) = \mathrm{int}\,\mathrm{dom}_H(c)$;
(c) $\mathrm{loc}_P(c) \subseteq \mathrm{int}\,\mathrm{dom}_P(c)$;
(d) $\mathrm{loc}_V(c) \cup \mathrm{loc}_H(c) = \mathrm{loc}_P(c)$;
(e) $\mathrm{loc}_V(c) \cap \mathrm{loc}_H(c) = \mathrm{int}\,\mathrm{dom}_S(c) = \mathrm{int}\,\mathrm{dom}_{CS}(c)$. ■

We say that c is *locally X-subordinate* in case $\mathrm{loc}_X(c) = \mathrm{dom}(c)$. From Lemma 1.3 it follows immediately

1.4. COROLLARY. *If $X \in \{V, H, P\}$, then every smooth locally X-subordinate curve in $\mathbb{R}^2$ is X-directed. Conversely, if $X \in \{V, H\}$, then every X-directed curve in $\mathbb{R}^2$ is locally X-subordinate.* ■

Throughout this paper in several constructions we use the following real smooth function ϑ on $\mathbb{R}$ defined by

$$\vartheta(t) = \begin{cases} 0 & \text{for } t \le 0 \\ \exp(-1/t) & \text{for } t > 0. \end{cases} \tag{1.1}$$

The following example shows that a P-directed curve need not be locally P-subordinate, which means that the inclusion in condition (c) of Lemma 1.3 is essential.

1.5. EXAMPLE. Let $c : \mathbb{R} \to \mathbb{R}^2$ be a smooth curve defined by

$$c(t) = \begin{cases} (\vartheta(-t), 0) & \text{for } t < 0 \\ (0,0) & \text{for } t = 0 \\ (0, \vartheta(t)) & \text{for } t > 0. \end{cases}$$

Clearly, c is P-directed but not locally P-subordinate at 0. ∎

Let $X \in \{V, H, P\}$. A curve $c : I \to \mathbb{R}^2$ is called *globally X-subordinate* if there is an X-principal line L such that $c(I) \subseteq L$. It is easy to verify

1.6. PROPOSITION. *If $X \in \{V, H\}$ and c is a smooth curve in $\mathbb{R}^2$, then the following conditions are equivalent*:

(a) *c is X-directed*;
(b) *c is locally X-subordinate*;
(c) *c is globally X-subordinate.* ∎

Note that from Corollary 1.2 and Proposition 1.6 we get

1.7. COROLLARY. *If c is a P-directed curve in $\mathbb{R}^2$, then* $\mathrm{dom}_R(c) \subseteq \mathrm{loc}_P(c)$. *More precisely, c restricted to any connected component of* $\mathrm{dom}_R(c)$ *is globally P-subordinate, so every regular P-directed curve is globally P-subordinate.* ∎

Obviously, every globally P-subordinate curve is locally P-subordinate, but conversely this need not be satisfied.

1.8. EXAMPLE. Let $c : \mathbb{R} \to \mathbb{R}^2$ be a smooth curve defined by

$$c(t) = \begin{cases} ((\vartheta(-1-t), 0) & \text{for } t < -1 \\ (0,0) & \text{for } -1 \le t \le 0 \\ (0, \vartheta(t)) & \text{for } t > 0. \end{cases}$$

Clearly, c is locally P-subordinate. Moreover, observe that the image of c is contained in $(\mathbb{R} \times \{0\}) \cup (\{0\} \times \mathbb{R})$ but it is not contained in $\mathbb{R} \times \{0\}$ or $\{0\} \times \mathbb{R}$, separately. ∎

Let c be a smooth curve in $\mathbb{R}^2$. By a *nonsingular parameter* of c we shall mean any element of the set

$$\mathrm{dom}_{NS}(c) = \mathrm{dom}(c) \setminus \mathrm{dom}_{CS}(c).$$

We say that c is *nonsingular* (*almost regular*) if $\mathrm{dom}_{CS}(c) = \emptyset$ ($\mathrm{int}\,\mathrm{dom}_{CS}(c) = \emptyset$). Since $\mathrm{int}\,\mathrm{dom}_S(c) = \mathrm{int}\,\mathrm{dom}_{CS}(c)$ by Lemma 1.3, c is almost regular if and only if $\mathrm{int}\,\mathrm{dom}_S(c) = \emptyset$, i.e. $\mathrm{cl}\,\mathrm{dom}_R(c) = \mathrm{dom}(c)$. Lemma 1.3 and Proposition 1.6 imply

1.9. PROPOSITION. *Every nonsingular (almost regular) locally P-subordinate curve in $\mathbb{R}^2$ is globally P-subordinate.* ∎

1.10. LEMMA. *Every nonsingular P-directed curve in $\mathbb{R}^2$ is locally P-subordinate.*

Proof. Let c be a nonsingular P-directed curve in $\mathbb{R}^2$ and let $s \in \mathrm{dom}(c)$. First, if s is regular, there is a neighbourhood U of s such that the curve $c' = c|U$ is regular and P-directed. Then, by Corollary 1.7, c' is globally P-subordinate, which means that c is locally P-subordinate at s.

Suppose now that s is not regular, which means that $s \in \mathrm{dom}_S(c) \cap \mathrm{dom}_{NS}(c)$ because c is nonsingular. Without loss of generality, we can assume further that $s = 0$. Clearly, there is the least positive integer $k \geq 1$ such that $(D^k\dot{c})(0) \neq 0$. Thus, if $c = (\alpha, \beta)$, then $(D^k\dot{c})(0) = [(D^k\dot{\alpha})(0), (D^k\dot{\beta})(0)] \neq 0$. We can assume that $(D^k\dot{\alpha})(0) \neq 0$, so there is $\varepsilon > 0$ such that

$$(D^k\dot{\alpha})(t) \neq 0 \text{ for } t \in U = (-\varepsilon, \varepsilon).$$

Since $(D^i\dot{\alpha})(0) = 0$ for $i < k$, by the Taylor formula we have

$$\dot{\alpha}(t) = \frac{(D^k\dot{\alpha})(\theta t)}{k!} t^k \text{ for } t \in U$$

where $\theta = \theta(t) \in (0;1)$, whence $\dot{\alpha}(t) \neq 0$ for $t \in U\setminus\{0\}$. Therefore and since c is P-directed, we have $\dot{\beta}(t) = 0$ for $t \in$ U. Thus, $c|U$ is globally V-subordinate, so $0 \in \mathrm{loc}_P(c)$. ∎

Clearly, this Lemma and Proposition 1.9 imply

1.11. PROPOSITION. *Every nonsingular P-directed curve in $\mathbb{R}^2$ is globally P-subordinate.* ∎

Note that this proposition implies the following corollary being a generalization of Corollary 1.7.

1.12. COROLLARY. *If c is a P-directed curve in $\mathbb{R}^2$, then $\mathrm{dom}_{NS}(c) \subseteq \mathrm{loc}_P(c)$. More precisely, c restricted to any connected component of $\mathrm{dom}_{NS}(c)$ is globally P-subordinate.* ∎

Applying this Corollary and Proposition 1.6 one has

1.13. PROPOSITION. *If $X \in \{V, H, P\}$ and c is a smooth curve in $\mathbb{R}^2$, then the following conditions are equivalent:*

(a) $\mathrm{dom}(c) = \mathrm{dom}_X(c)$;
(b) $\mathrm{dom}_{NS}(c) \subseteq \mathrm{dom}_X(c)$;
(c) $\mathrm{dom}_R(c) \subseteq \mathrm{dom}_X(c)$;
(d) $\mathrm{dom}_{NS}(c) \subseteq \mathrm{loc}_X(c)$;
(e) $\mathrm{dom}_R(c) \subseteq \mathrm{loc}_X(c)$. ■

Remark that since condition (a) of Proposition 1.13 means that c is X-directed, we can regard the other ones as characterizations of X-directed curves among smooth curves in $\mathbb{R}^2$.

For any $a, b \in \mathbb{R}$, we put $V_a = \{a\} \times \mathbb{R}$ and $H_b = \mathbb{R} \times \{b\}$. By a *principal cross* we shall mean a subset K of $\mathbb{R}^2$ of the form $\mathbb{K}_p = V_a \cup H_b$ where $p = (a, b)$ is called the *origin* of K. The principal cross $\mathbb{K} = \mathbb{K}_o$ with origin $o = (0, 0)$ will also be called the *central principal* one. A curve c in $\mathbb{R}^2$ is called *locally K-subordinate* at a parameter s if there are a neighbourhood U of s in $\mathrm{dom}(c)$ and a principal cross K such that $c(U) \subseteq K$. The set of all such parameters will be denoted by $\mathrm{loc}_K(c)$. We say that c is *locally K-subordinate* provided that $\mathrm{loc}_K(c) = \mathrm{dom}(c)$. Clearly, every locally P-subordinate curve is locally K-subordinate, but not conversely in general. However, every smooth locally K-subordinate curve is P-directed. By an easy verification we get

1.14. Proposition. *If $X \in \{P, K\}$ and if c is a P-directed curve in $\mathbb{R}^2$, then the following conditions are equivalent*:

(a) $\mathrm{dom}(c) = \mathrm{loc}_X(c)$;
(b) $\mathrm{dom}_S(c) \subseteq \mathrm{loc}_X(c)$;
(c) $\mathrm{dom}_{CS}(c) \subseteq \mathrm{loc}_X(c)$. ■

One can see that this proposition can be false in the case when $X \in \{V, H\}$. For example, the horizontal curve $c = (\mathrm{id}_{\mathbb{R}}, 0) : \mathbb{R} \to \mathbb{R}^2$ satisfies $\mathrm{dom}_S(c) = \mathrm{dom}_{CS}(c) = \mathrm{loc}_V(c) = \emptyset$ but $\mathrm{dom}(c) = \mathbb{R} \neq \mathrm{loc}_V(c) = \emptyset$. However, we have

1.15. Proposition. *If $X \in \{V, H\}$ and if c is a P-directed curve in $\mathbb{R}^2$, then the following conditions are equivalent*:

(a) $\mathrm{dom}(c) = \mathrm{loc}_X(c)$;
(b) $\mathrm{dom}_S(c) \subseteq \mathrm{loc}_X(c) \neq \emptyset$;
(c) $\mathrm{dom}_{CS}(c) \subseteq \mathrm{loc}_X(c) \neq \emptyset$.

Proof. Since the cases $X = V$ and $X = H$ are completely analogous, we can assume further that $X = V$. Obviously, it remains to prove the implication (c) $\Rightarrow$ (a), or equivalently, the statements (1) and (2) below.

(1) *If* $\emptyset = \mathrm{dom}_{CS}(c) \subseteq \mathrm{loc}_V(c) \neq \emptyset$, *then* $\mathrm{dom}(c) = \mathrm{loc}_V(c)$.

Indeed, we have $\mathrm{dom}_{CS}(c) \subseteq \mathrm{loc}_V(c) \subseteq \mathrm{loc}_P(c)$ and from Proposition 1.14 for $X = P$ it follows that $\mathrm{dom}(c) = \mathrm{loc}_P(c)$, and so, $\mathrm{loc}_V(c) \cup \mathrm{loc}_H(c) = \mathrm{dom}(c)$ by Lemma 1.3(d). Furthermore, from condition (e) of this lemma we get $\mathrm{loc}_V(c) \cap \mathrm{loc}_H(c) = \emptyset$. Therefore, since $\mathrm{loc}_V(c)$ and $\mathrm{loc}_H(c)$ are open in the connected space $\mathrm{dom}(c)$ and $\mathrm{loc}_V(c) \neq \emptyset$, we have $\mathrm{dom}(c) = \mathrm{loc}_V(c)$.

(2) *If* $\emptyset \neq \mathrm{dom}_{CS}(c) \subseteq \mathrm{loc}_V(c)$, *then* $\mathrm{dom}(c) = \mathrm{loc}_V(c)$.

Observe first that this statement is trivial in the case when $\mathrm{dom}_{CS}(c) = \mathrm{dom}(c)$. Therefore, we can assume further that $\mathrm{dom}_{CS}(c) \neq \mathrm{dom}(c)$, i.e. $\mathrm{dom}_{NS}(c) \neq \emptyset$. Let us take a parameter $s \in \mathrm{dom}_{NS}(c)$. Consider the sets $F^- = \{t \in \mathrm{dom}_{CS}(c) : t < s\}$ and $F^+ = \{t \in \mathrm{dom}_{CS}(c) : t > s\}$. Note that F^- and F^+ are closed disjoint subsets of $\mathrm{dom}(c)$ such that $F^- \cup F^+ = \mathrm{dom}_{CS}(c)$. Without loss of generality we can assume that $F^- \neq \emptyset$. Let us set $t^- = \max F^-$ and $t^+ = \min F^+$ if $F^+ \neq \emptyset$. Consider the interval I defined to be $(t^-; t^+)$ if $F^+ \neq \emptyset$ and $(t^-; +\infty) \cap \mathrm{dom}(c)$ if $F^+ = \emptyset$. Of course, $I \cap \mathrm{dom}_{CS}(c) = \emptyset$ and I is a neighbourhood of s in $\mathrm{dom}(c)$. Let $d = c|I$ and note that d is a P-directed curve such that $\mathrm{dom}_{CS}(d) = \emptyset$. Moreover, observe that $\mathrm{loc}_V(d) \neq \emptyset$ because $\mathrm{loc}_V(d) = \mathrm{loc}_V(c) \cap I$ and $t^- \in \mathrm{dom}_{CS}(c) \subseteq \mathrm{loc}_V(c)$, which means that d satisfies the assumption of statement (1). Therefore, by statement (1) we have $\mathrm{dom}(d) = \mathrm{loc}_V(d)$, whence $s \in \mathrm{loc}_V(d) \subseteq \mathrm{loc}_V(c)$ and since s can be an arbitrary point of $\mathrm{dom}_{NS}(c)$, we conclude that $\mathrm{dom}_{NS}(c) \subseteq \mathrm{loc}_V(c)$. Thus and since $\mathrm{dom}(c) = \mathrm{dom}_{CS}(c) \cup \mathrm{dom}_{NS}(c)$ and $\mathrm{dom}_{CS}(c) \subseteq \mathrm{loc}_V(c)$, it follows that $\mathrm{dom}(c) = \mathrm{loc}_V(c)$.

To sum up we have proved the statements (1) and (2) which are equivalent to the implication (c) $\Rightarrow$ (a). ■

Let c be a P-directed curve in $\mathbb{R}^2$. Let us set

$$\mathrm{dom}_{PS}(c) = \mathrm{dom}(c) \setminus \mathrm{loc}_P(c)$$

and note that $\mathrm{dom}_{PS}(c)$ is a closed subset of $\mathrm{dom}(c)$. Moreover, from Corollary 1.12 it follows that $\mathrm{dom}_{PS}(c) \subseteq \mathrm{dom}_{CS}(c)$. Obviously, by Proposition 1.14 we get

1.16. Corollary. *If c is a P-directed curve in $\mathbb{R}^2$, then the following conditions are equivalent:*

(a) $\mathrm{dom}(c) = \mathrm{loc}_K(c)$;
(b) $\mathrm{dom}_S(c) \subseteq \mathrm{loc}_K(c)$;
(c) $\mathrm{dom}_{CS}(c) \subseteq \mathrm{loc}_K(c)$;
(d) $\mathrm{dom}_{PS}(c) \subseteq \mathrm{loc}_K(c)$. ■

It is easy to verify

1.17. Corollary. *Under the same assumptions, if* $\mathrm{dom}_{PS}(c)$ *is discrete in* $\mathrm{dom}(c)$, *then* c *is locally* K*-subordinate.* ■

2. A characterization of smooth curves in $\mathbb{R} \times_k \mathbb{R}$

We shall regard $\mathbb{R}^2$ as a real normed vector space under the coordinatewise operations and the norm $||p|| = (x^2+y^2)^{1/2}$ for $p = (x, y)$. In particular, $\mathbb{R}^2$ will be regarded as a topological space under the Euclidean topology. For any $p \in \mathbb{R}^2$ denote by τ_p the *translation* of $\mathbb{R}^2$ via p, i.e. $\tau_p(x) = x + \mathrm{p}$. If $A \subseteq \mathbb{R}^2$, we set $A + p = \tau_p(A)$. Clearly, $\mathbb{K}_p = \mathbb{K} + p$ for $p \in \mathbb{R}^2$.

Let A be a subset of $\mathbb{R}^2$. We shall regard A as a differential space with structure $C^\infty(A)$ of all real smooth functions on A. Clearly, A is a differential space of class $\mathcal{D}_0$ (see [4], Theorem (2.1)). For any $x \in A$ denote by T_xA the tangent vector space of A at x. We associate with A the dimension function $\delta_A : A \to \mathbb{Z}^+$ defined by $\delta_A(x) = \dim T_x\mathrm{A}$. It is well known that δ_A is upper semicontinuous (see [1], Corollary 1). A point p of A is called *regular* (*singular*) if δ_A is continuous (discontinuous) at p, or equivalently, constant (nonconstant) locally at p. Moreover, it is also known that the set A^* (sing A) of all regular (singular) points of A is an open (closed) and dense (boundary) subset of A (see [1], Corollary 3). We set

$$A^i = \{p \in A : \delta_A(p) = i\} \text{ for } i = 0, 1, 2.$$

Clearly, A^0, A^1 and A^2 are disjoint and $A^0 \cup A^1 \cup A^2 = A$. Since δ_A is upper semicontinuous, it follows that A^0 and $A^0 \cup A^1$ are open subsets of A and A^2 is a closed subset of A. Moreover, it is known that A^0 consists of all isolated points of A (see [1], Proposition 2), so A^0 is a discrete subset of A.

Let $p \in A \subseteq \mathbb{R}^2$. If $X \in \{V, H, P\}$, we say that A is *locally* X*-subordinate* at p in case there are a neighbourhood U of p in $\mathbb{R}^2$ and an X-principal line L such that $A \cap U \subseteq L$. Moreover, we say that A is *locally* K*-subordinate* at p if there is a neighbourhood U of p in $\mathbb{R}^2$ such that $A \cap U \subseteq \mathbb{K}_p$. For $X \in \{V, H, P, K\}$ we denote by $\mathrm{loc}_X A$ the set of all points p of A such that A is locally X-subordinate at p. Clearly, $\mathrm{loc}_X A$ is an open subset of A. We call A *locally* X*-subordinate* in case $\mathrm{loc}_X A = A$. If $X \in \{V, H, P\}$ and A is contained in an X-principal line, we call A *globally* X*-subordinate.* Similarly, if A is contained in a principal cross, we say that it is *globally* K*-subordinate.* Obviously, we have the following lemmas.

2.1. Lemma. *For any subset* A *of* $\mathbb{R}^2$ *the following conditions hold:*

(a) $\mathrm{loc}_V A \cap \mathrm{loc}_H A = A^0$;
(b) $\mathrm{loc}_V A \cup \mathrm{loc}_H A = \mathrm{loc}_P A \subseteq A^0 \cup A^1$;
(c) $\mathrm{loc}_K A \setminus \mathrm{loc}_P A \subseteq A^2$. ■

2.2. LEMMA. *If A is a locally K-subordinate subset of $\mathbb{R}^2$, then the following conditions hold*:

(a) $A^1 = \mathrm{loc}_P A \setminus A^0$;
(b) $A^2 = \mathrm{loc}_K A \setminus \mathrm{loc}_P A \subseteq \mathrm{sing} A$ *and* A^2 *is a discrete subset of* A. ■

It is easy to verify

2.3. PROPOSITION. *Let A be a connected subset of $\mathbb{R}^2$.*
(1) *If $X \in \{V, H, P\}$ and A is locally X-subordinate, then A is globally X-subordinate*;
(2) *If A is locally K-subordinate, then* $\mathrm{loc}_P A = A^*$. ■

By applying Lemma 2.2 we get

2.4. PROPOSITION. *If A is a nonsingle connected locally K-subordinate subset of $\mathbb{R}^2$, then the following conditions hold*:
(a) $A^0 = \emptyset$;
(b) $A^1 = \mathrm{loc}_P A = A^*$;
(c) $A^2 = \mathrm{sing} A$. ■

From this Proposition and Lemma 2.2 we obviously get

2.5. COROLLARY. *If A is a connected locally K-subordinate subset of $\mathbb{R}^2$, then the following statements hold*:
(1) $\mathrm{sing} A$ *is a discrete closed subset of* A;
(2) *If A is closed in $\mathbb{R}^2$, then* $\mathrm{sing} A$ *is a discrete closed subset of* $\mathbb{R}^2$;
(3) *If A is compact, then* $\mathrm{sing} A$ *is finite.* ■

The following example shows that if A is a disconnected and compact locally H-subordinate subset of $\mathbb{R}^2$, then the set $\mathrm{sing} A$ can be dense in itself and of the continuum power.

2.6. EXAMPLE. Consider the closed interval $I = [0; 1] \subseteq \mathbb{R}$. Let C be the Cantor set regarded as a subset of I, that is, C consists of all $x \in I$ which have the following representations

$$x = \sum_{i=1}^{\infty} \xi_i 3^{-i}$$

where $\xi_i = 0, 2$. It is known that C is dense in itself and compact boundary subset of I of the continuum power. Clearly,

$$I \setminus C = \bigcup_{n=1}^{\infty} (a_n; b_n)$$

where $\{(a_n; b_n) : n \in \mathbb{N}\}$ is a family of disjoint open intervals of $\mathbb{R}$. For any $n \in \mathbb{N}$ let us take a discrete countable subset P_n of $(a_n; b_n)$ such that $a_n, b_n \in \overline{P}_n$. Consider the set

$$A = \bigcup_{n=1}^{\infty} P_n \cup C.$$

We can regard that $A \subseteq \mathbb{R}^2$ via the identification $x \mapsto (x, 0)$. Clearly, A is a disconnected and compact globally H-subordinate subset of $\mathbb{R}^2$. Moreover, note that

$$A^0 = A^* = \bigcup_{n=1}^{\infty} P_n, \quad A^1 = \operatorname{sing} A = C \quad \text{and} \quad A^2 = \emptyset. \blacksquare$$

Let $X \in \{V, H, P, K\}$. Denote by $\operatorname{lso}(X)$ the class of all locally X-subordinate subsets of $\mathbb{R}^2$. It is easy to verify

2.7. Proposition. *Let $X \in \{V, H, P, K\}$. The class $\operatorname{lso}(X)$ has the following properties:*

(1) *If $A \in \operatorname{lso}(X)$ and $B \subseteq A$, then $B \in \operatorname{lso}(X)$;*

(2) *If $A \subseteq \mathbb{R}^2$ and for each $p \in A$ there is a neighbourhood U of p in $\mathbb{R}^2$ such that $A \cap U \in \operatorname{lso}(X)$, then $A \in \operatorname{lso}(X)$;*

(3) *If in addition $X \in \{V, H, K\}$, then $A, B \in \operatorname{lso}(X)$ involves $A \cup B \in \operatorname{lso}(X)$.* $\blacksquare$

The proposition above immediately implies

2.8. Corollary. *Let $X \in \{V, H, K\}$. If $\mathfrak{F}$ is a locally finite family of sets from the class $\operatorname{lso}(X)$, then $\bigcup \mathfrak{F} \in \operatorname{lso}(X)$.* $\blacksquare$

The following example shows that the union of a countable family of sets from the class $\operatorname{lso}(K)$ as well as the closure of a set of this class need not belong to $\operatorname{lso}(K)$. Analogous examples we can construct for $X \in \{V, H, P\}$.

2.9. Example. Let us set $\mathbb{K}_n = \mathbb{K}_{(2^{-n}, 2^{-n})}$ for $n \in \mathbb{N}$. Consider the families $\mathfrak{F}_1 = \{\mathbb{K}_n : n \in \mathbb{N}\}$ and $\mathfrak{F}_0 = \{\mathbb{K}\} \cup \mathfrak{F}_1$. Clearly, $\mathfrak{F}_1$ is a locally finite family of sets from $\operatorname{lso}(K)$, so $\bigcup \mathfrak{F}_1 \in \operatorname{lso}(K)$ by Corollary 2.8. On the other hand, $\mathfrak{F}_0$ is not such a family and $\bigcup \mathfrak{F}_0 \notin \operatorname{lso}(K)$ because $\bigcup \mathfrak{F}_0$ is not locally K-subordinate at o. Finally, note that $\bigcup \mathfrak{F}_0$ is the closure of $\bigcup \mathfrak{F}_1$. $\blacksquare$

If A is a subset of $\mathbb{R} \times_k \mathbb{R}$ ($A \subseteq \mathbb{R}^2$), then by $\mathcal{S}^k(A)$ will be denoted the differential structure on A induced from $\mathbb{R} \times_k \mathbb{R}$. We say that A is a *C^∞ subset* of $\mathbb{R} \times_k \mathbb{R}$ in case $\mathcal{S}^k(A) = C^\infty(A)$. Obviously, every vertical or horizontal line in $\mathbb{R} \times_k \mathbb{R}$ is such a subset. Let us denote by $\mathrm{sub}^\infty(\mathbb{R} \times_k \mathbb{R})$ the class of all C^∞ subsets of $\mathbb{R} \times_k \mathbb{R}$. We need the following lemmas.

2.10. LEMMA. (see [2], Lemma 2.1). *For every $p \in \mathbb{R}^2$ the translation τ_p is a diffeomorphism of $\mathbb{R} \times_k \mathbb{R}$, and so, the family $\mathrm{sub}^\infty(\mathbb{R} \times_k \mathbb{R})$ is invariant under any translation of $\mathbb{R}^2$.* ∎

Let us set $\mathbb{R}^2_o = \mathbb{R}^2 \setminus \{o\}$ where $o = (0,0)$.

2.11. LEMMA (see [2], Corollary 1.4). *Assume that $\alpha \in \mathcal{F}^k$. Then $\alpha \in \mathcal{S}^k$ if and only if $\alpha|\mathbb{R}^2_o \in \mathcal{S}^k(\mathbb{R}^2_o)$ and $\alpha|\mathbb{K} \in C^\infty(\mathbb{K})$.* ∎

By an easy verification we get

2.12. LEMMA. *The following properties hold:*

(1) *If $A \in \mathrm{sub}^\infty(\mathbb{R} \times_k \mathbb{R})$ and $B \subseteq A$, then $B \in \mathrm{sub}^\infty(\mathbb{R} \times_k \mathbb{R})$.*

(2) *If $A \subseteq \mathbb{R}^2$ and for each $p \in A$ there is an open neighbourhood U of p in $\mathbb{R}^2$ such that $A \cap U \in \mathrm{sub}^\infty(\mathbb{R} \times_k \mathbb{R})$, then $A \in \mathrm{sub}^\infty(\mathbb{R} \times_k \mathbb{R})$.* ∎

2.13. LEMMA. *For any $p \in \mathbb{R}^2$ we have $\mathbb{K}_p \in \mathrm{sub}^\infty(\mathbb{R} \times_k \mathbb{R})$.*

Proof. By Lemma 2.10 and since $\mathbb{K}_p = \mathbb{K} + p$ for $p \in \mathbb{R}^2$, it suffices to show that the central principal cross $\mathbb{K}$ is a C^∞ subset of $\mathbb{R} \times_k \mathbb{R}$. Since $\mathbb{K}$ equipped with the structure $\mathcal{S}^k(\mathbb{K})$ is a differential subspace of $\mathbb{R} \times_k \mathbb{R}$, we conclude that the differential structure $\mathcal{S}^k(\mathbb{K})$ is generated by the restrictions $\varphi|\mathbb{K}$ for $\varphi \in \mathcal{S}^k$. From the definition of $\mathcal{S}^k$ it follows that for any $\varphi \in \mathcal{S}^k$ we have $\varphi \circ i \in C^\infty(\mathbb{R})$ and $\varphi \circ j \in C^\infty(\mathbb{R})$ where $i(x) = (x, 0)$ and $j(y) = (0, y)$ for $x, y \in \mathbb{R}$. It is seen that if $\varphi' = \varphi \circ i$ and $\varphi'' = \varphi \circ j$, then the function $\widetilde{\varphi} : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\widetilde{\varphi}(x,y) = \varphi'(x) + \varphi''(y) - \varphi(o)$$

belongs to $C^\infty(\mathbb{R}^2)$. Note that $\widetilde{\varphi}|\mathbb{K} = \varphi|\mathbb{K}$, which means that the differential structure on $\mathbb{K}$ is generated by restrictions of smooth functions on $\mathbb{R}^2$, and so, $\mathbb{K}$ is a C^∞ subset of $\mathbb{R} \times_k \mathbb{R}$. ∎

2.14. LEMMA. *Let $\{p_n\}$ be an infinite sequence of distinct points of $\mathbb{R}^2$ such that $\lim p_n = o$ and $p_n \notin \mathbb{K}$ for each $n \in \mathbb{N}$. If $\{t_n\}$ is an infinite sequence of real numbers such that $\lim t_n = 0$, then there is a function $\varphi \in \mathcal{S}^2$ such that $\varphi|\mathbb{K} = 0$ and $\varphi(p_n) = t_n$ for each $n \in \mathbb{N}$.*

Proof. One can see that there is a discrete sequence $\{U_n\}$ of open subsets of $\mathbb{R}^2_o$ such that $p_n \in U_n$ and $\overline{U}_n \cap \mathbb{K} = \emptyset$ for each $n \in \mathbb{N}$. Next, we can choose a sequence $\{\varphi_n\}$ of real smooth functions on $\mathbb{R}^2$ such that

$0 \leq \varphi_n(q) \leq 1$ for $q \in \mathbb{R}^2$, $\varphi_n(p_n) = 1$ and supp $\varphi_n \subseteq U_n$. Define the function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ by

$$\varphi(q) = \sum_{n=1}^{\infty} t_n \varphi_n(q).$$

Clearly, φ is continuous such that $\varphi|\mathbb{K} = 0$ and $\varphi(p_n) = t_n$ for $n \in \mathbb{N}$. Moreover, $\varphi|\mathbb{R}^2_o \in C^\infty(\mathbb{R}^2_o)$, so $\varphi \in \mathcal{S}^2$ by Lemma 2.11. ∎

2.15. THEOREM. $\text{sub}^\infty(\mathbb{R} \times_1 \mathbb{R}) = \text{sub}^\infty(\mathbb{R} \times_2 \mathbb{R}) = \text{lso}(K)$.

Proof. Clearly, the inclusion $\text{lso}(K) \subseteq \text{sub}^\infty(\mathbb{R} \times_k \mathbb{R})$ follows from Lemmas 2.12 and 2.13. To prove the converse inclusion, suppose to the contrary that $A \in \text{sub}^\infty(\mathbb{R} \times_k \mathbb{R}) \backslash \text{lso}(K)$. This means that there is $p \in A$ such that A is not locally K-subordinate at p. By Lemma 2.10 we can assume that $p = o$. It follows that for every open neighbourhood U of o in $\mathbb{R}^2$ we have $(A \cap U) \backslash \mathbb{K} \neq \emptyset$. This implies that there is a sequence $\{p_n\}$ of distinct points of A such that $\lim p_n = o$ and $p_n \notin \mathbb{K}$ for $n \in \mathbb{N}$. Let us set $t_n = \|p_n\|^{1/2}$ for $n \in \mathbb{N}$. By Lemma 2.14 there is a function $\varphi \in \mathcal{S}^2 \subseteq \mathcal{S}^k$ such that $\varphi(o) = 0$ and $\varphi(p_n) = \|p_n\|^{1/2}$.

We set $\varphi' = \varphi|A$ and note that $\varphi' \in \mathcal{S}^k(A)$. Since A is a C^∞ subset of $\mathbb{R} \times_k \mathbb{R}$, it follows that $\varphi' \in C^\infty(A)$. Then there are an open neighbourhood U of o in $\mathbb{R}^2$ and a function $\psi \in C^\infty(\mathbb{R}^2)$ such that $\psi|A \cap U = \varphi'|A \cap U$. Clearly, $\psi(o) = \varphi(o) = 0$ and since $\lim p_n = o$, we have $\psi(p_n) = \varphi(p_n) = \|p_n\|^{1/2}$ for sufficiently large n. Hence we get

$$\lim \frac{\psi(p_n) - \psi(o)}{\|p_n\|} = \lim \|p_n\|^{-1/2} = \infty,$$

which means that the function ψ is not differentiable at o, a contradiction. This completes the proof of our assertion. ∎

A family $\Sigma \subseteq \text{lso}(K)$ is called a C^∞ *generator* of $\text{lso}(K)$ in case the following condition holds:

If $\alpha \in \mathcal{F}^1$ and $\alpha|A \in C^\infty(A)$ for each $A \in \Sigma$, then $\alpha|B \in C^\infty(B)$ for all $B \in \text{lso}(K)$.

It is seen that the families of all principal lines and of all principal crosses are C^∞ generators of $\text{lso}(K)$. Moreover, if Σ is a C^∞ generator of $\text{lso}(K)$, then so is the family $\Sigma(\mathfrak{B}) = \{A \cap U : A \in \Sigma,\ U \in \mathfrak{B}\}$ where $\mathfrak{B}$ is an arbitrary topological base of $\mathbb{R}^2$. We obviously have

2.16. LEMMA. *If Σ is a C^∞ generator of $\text{lso}(K)$, then*

$$\mathcal{S}^k = \{\alpha \in \mathcal{F}^k : \alpha|A \in C^\infty(A)\ \forall A \in \Sigma\}. \ ∎$$

It is easy to verify

2.17. PROPOSITION. *Let M be a differential space. Let Σ be a C^∞ generator of* $\mathrm{Iso}(K)$. *A map (continuous map) $f : \mathbb{R}^2 \to M$ is smooth from $\mathbb{R} \times_1 \mathbb{R}$ ($\mathbb{R} \times_2 \mathbb{R}$) to M if and only if $f|A : A \to M$ is smooth for each $A \in \Sigma$.* ■

We say that a curve $c : I \to \mathbb{R}^2$ is *locally C^∞ subordinate to $\mathbb{R} \times_k \mathbb{R}$* if for each $s \in I$ there is an open neighbourhood U of s in I such that $c(U) \in \mathrm{sub}^\infty(\mathbb{R} \times_k \mathbb{R})$. From Theorem 2.15 we obviously get

2.18. COROLLARY. *A curve c in $\mathbb{R}^2$ is locally K-subordinate if and only if it is locally C^∞ subordinate to $\mathbb{R} \times_k \mathbb{R}$.* ■

Clearly, we have

2.19. LEMMA. *Let c be a smooth curve in $\mathbb{R}^2$. If c is locally C^∞ subordinate to $\mathbb{R} \times_k \mathbb{R}$, then it is smooth in $\mathbb{R} \times_k \mathbb{R}$.* ■

2.20. THEOREM. *If c is a smooth curve in $\mathbb{R}^2$, then the following conditions are equivalent:*

(a) *c is smooth in $\mathbb{R} \times_k \mathbb{R}$;*
(b) *c is locally K-subordinate;*
(c) *c is locally C^∞ subordinate to $\mathbb{R} \times_k \mathbb{R}$.*

Proof. From Corollary 2.18 and Lemma 2.19 it follows that the implications (b)⇒(c) and (c)⇒(a) are satisfied. Thus, it remains to prove the implication (a)⇒(b). Suppose to the contrary that there exists a smooth curve $c : I \to \mathbb{R} \times_k \mathbb{R}$ which is not locally K-subordinate. This means that there is a parameter $s \in I$ such that for each $\varepsilon > 0$ we have

$$c([s-\varepsilon; s+\varepsilon] \cap I) \cap (\mathbb{R}^2 \backslash \mathbb{K}_p) \neq \emptyset$$

where $p = c(s)$. By Lemma 2.10 and since the parameterization of c may be changed, one can assume that $s = 0$ and $p = o$. Then there is a sequence $\{t_n\}$ of parameters of c converging to 0 such that the sequence $\{c(t_n)\}$ consists of distinct points of $\mathbb{R}^2 \backslash \mathbb{K}$ and $\lim c(t_n) = o$. By Lemma 2.14 there is a function $\varphi \in S^k$ such that $\varphi(c(t_n)) = (-1)^n \cdot t_n$ and $\varphi(c(0)) = 0$. Hence we get $(\varphi(c(t_n)) - \varphi(c(0)))/t_n = (-1)^n$, which means that c is not smooth curve in $\mathbb{R} \times_k \mathbb{R}$ at 0, a contradiction. ■

Denote by $\mathrm{cur}(\mathbb{R} \times_k \mathbb{R})$ the class of all smooth curves in $\mathbb{R} \times_k \mathbb{R}$. In turn, by $\mathrm{cur}(K)$ ($\mathrm{cur}^\infty(K)$) we denote the class of all locally K-subordinate curves (smooth curves) in $\mathbb{R}^2$. Clearly, Theorem 2.20 implies

2.21. COROLLARY. $\mathrm{cur}(\mathbb{R} \times_1 \mathbb{R}) = \mathrm{cur}(\mathbb{R} \times_2 \mathbb{R}) = \mathrm{cur}^\infty(K)$. ■

By a *principal K-graph* in $\mathbb{R}^2$ we shall mean a compact connected locally K-subordinate subset of $\mathbb{R}^2$. The simplest example of such a graph is given

by a principal closed segment, i.e. a closed segment lying in a principal line. One can see that every principal K-graph is a union of a finite family of principal closed segments. From Theorem 2.20 it follows that if c is a smooth curve in $\mathbb{R} \times_k \mathbb{R}$, then for any $a, b \in \mathrm{dom}(c)$ such that $a \leq b$ the image $c([a;b])$ is a principal K-graph.

The following example shows that a P-directed curve in $\mathbb{R}^2$ need not be locally K-subordinate, i.e. smooth in $\mathbb{R} \times_k \mathbb{R}$.

2.22. Example. Clearly, one can construct functions $\alpha', \beta' \in C^\infty([0;1])$ satisfying the following conditions:

$$\alpha'(t) > 0 \textit{ and } \beta'(t) = 0 \textit{ if } \frac{1}{k+1} < t < \frac{1}{k} \textit{ where } k \textit{ is odd};$$

$$\alpha'(t) = 0 \textit{ and } \beta'(t) > 0 \textit{ if } \frac{1}{k+1} < t < \frac{1}{k} \textit{ where } k \textit{ is even};$$

$$\alpha'(t) = \beta'(t) = 0 \textit{ if } t = \frac{1}{k}\ (k \in \mathbb{N}) \textit{ or } t = 0.$$

Let us set

$$\alpha(s) = \int_0^s \alpha'(t)\,dt \quad \text{and} \quad \beta(s) = \int_0^s \beta'(t)\,dt \text{ for } 0 \leq s \leq 1$$

and note that the curve $c = (\alpha, \beta) : [0;1] \to \mathbb{R}^2$ is P-directed. It is seen that we have the following decomposition:

$$\operatorname{im} c = \bigcup_{k=1}^{\infty} c(I_k) \cup \{o\} \text{ where } I_k = \left[\frac{1}{k+1}; \frac{1}{k}\right].$$

Clearly, $c(I_k)$ is a vertical (horizontal) closed segment in $\mathbb{R}^2$ provided that k is even (odd). One can see that this decomposition is unique in the following sense. If S is an arbitrary nonsingle segment in $\mathbb{R}^2$ such that $S \subseteq \operatorname{im} c$, then there is a unique $k \in \mathbb{N}$ such that $S \subseteq c(I_k)$. In addition, $\operatorname{im} c = c([0;1])$ is not a principal K-graph, so c is not smooth in $\mathbb{R} \times_k \mathbb{R}$. ∎

It is easy to verify

2.23. Lemma. *For any principal K-graph G in $\mathbb{R}^2$ there is a smooth curve $c : [0;1] \to \mathbb{R} \times_k \mathbb{R}$ such that $c([0;1]) = G$.* ∎

Denote by $\mathrm{gr}(K)$ the class of all principal K-graphs in $\mathbb{R}^2$. Clearly, $\mathrm{gr}(K)$ is a C^∞ generator of $\mathrm{lso}(K)$. If f is a smooth map from $\mathbb{R} \times_k \mathbb{R}$ to $\mathbb{R} \times_\ell \mathbb{R}$ where $k, \ell \in \{1,2\}$, then by Lemma 2.23 and Theorem 2.20 we conclude that $A \in \mathrm{gr}(K)$ involves $f(A) \in \mathrm{gr}(K)$. Denote by $\mathcal{S}^{k,\ell}$ the family of all smooth maps from $\mathbb{R} \times_k \mathbb{R}$ to $\mathbb{R} \times_\ell \mathbb{R}$. Moreover, we adopt that $\mathcal{F}^{1,1} = \mathcal{F}^{1,2}$ ($\mathcal{F}^{2,2}$) denotes the family of all maps (continuous maps) of $\mathbb{R}^2$. If c is a

curve in $\mathbb{R}^2$ and f is a map of $\mathbb{R}^2$, we set $f_\#(c) = f \circ c$. It is easy to verify

2.24. PROPOSITION. *Let $k, \ell \in \{1,2\}$, $(k,\ell) \neq (2,1)$, and $f \in \mathcal{F}^{k,\ell}$. Then the following conditions are equivalent:*

(a) $f \in \mathcal{S}^{k,\ell}$;

(b) $f_\#(\mathrm{cur}(\mathbb{R} \times_k \mathbb{R})) \subseteq \mathrm{cur}(\mathbb{R} \times_\ell \mathbb{R})$;

(c) $f_\#(\mathrm{cur}^\infty(K)) \subseteq \mathrm{cur}^\infty(K)$;

(d) *If $A \in \mathrm{gr}(K)$, then $f|A : A \to f(A)$ is a smooth map of C^∞ subsets of $\mathbb{R} \times_k \mathbb{R}$ and $\mathbb{R} \times_\ell \mathbb{R}$, respectively.* ∎

Clearly, this proposition implies equalities $\mathcal{S}^{1,2} = \mathcal{S}^{1,1}$ and $\mathcal{S}^{2,2} = \mathcal{S}^{1,1} \cap \mathcal{F}^{2,2}$. One can ask whether there is a corresponding characterization of smooth maps from $\mathbb{R} \times_2 \mathbb{R}$ to $\mathbb{R} \times_1 \mathbb{R}$. This problem has a solution for all cases $k, \ell \in \{1,2\}$ (Proposition 2.25). Let $\mathcal{C}^{k,\ell}$ denote the family of all continuous maps from $\mathbb{R} \times_k \mathbb{R}$ to $\mathbb{R} \times_\ell \mathbb{R}$ with respect to the corresponding Sikorski topologies. It is easy to verify

2.25. PROPOSITION. *Let $k, \ell \in \{1,2\}$ and $f \in \mathcal{C}^{k,\ell}$. Then the following conditions are equivalent:*

(a) $f \in \mathcal{S}^{k,\ell}$;

(b) $f_\#(\mathrm{cur}(\mathbb{R} \times_k \mathbb{R})) \subseteq \mathrm{cur}(\mathbb{R} \times_\ell \mathbb{R})$;

(c) $f_\#(\mathrm{cur}^\infty(K)) \subseteq \mathrm{cur}^\infty(K)$;

(d) *If $A \in \mathrm{gr}(K)$, then $f|A : A \to f(A)$ is a smooth map of C^∞ subsets of $\mathbb{R} \times_k \mathbb{R}$ and $\mathbb{R} \times_\ell \mathbb{R}$, respectively.* ∎

By the definitions $\mathcal{C}^{2,2} = \mathcal{F}^{2,2}$, so in the case when $(k,\ell) = (2,2)$ Propositions 2.24 and 2.25 coincide. However, for the remaining cases, the following question arises: what are functions belonging to $\mathcal{C}^{k,\ell}$. Since we do not know any full answer to this question, Proposition 2.25 is less useful than Proposition 2.24. One can observe that a reason of such a situation is also justified by the fact that there is still open the question: what is the kind of the Sikorski topology of $\mathbb{R} \times_1 \mathbb{R}$, called shortly the $\mathcal{S}^1$-topology, i.e. the weakest one on $\mathbb{R}^2$ for which all functions from $\mathcal{S}^1$ are continuous (see [2], Question 5.6). However, it is easily seen that if ϕ and ψ are functions from $\mathcal{S}^1$, then the function $(\phi, \psi) : \mathbb{R}^2 \to \mathbb{R}^2$ defined by the assignment $(x,y) \mapsto (\phi(x), \psi(y))$ belongs to $\mathcal{C}^{1,2}$ where ϕ and ψ can be taken from $\mathcal{S}^1 \setminus \mathcal{S}^2$ (see [2], Example 1.5). Moreover, since the $\mathcal{S}^1$-topology is stronger than the Euclidean one, it follows that $\mathcal{C}^{2,2} \subseteq \mathcal{C}^{1,2}$ and $\mathcal{C}^{2,1} \subseteq \mathcal{C}^{2,2}$. But it seems to be much more complicated to state anything about $\mathcal{C}^{1,1}$.

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ON A GROUP OF LINEAR TRANSFORMATIONS OF CIRCULAR DOMAIN IN $\mathbb{C}^n$

We consider properties of a group of complex linear transformations of a bounded circular domain in $\mathbb{C}^n$. It is a compact not semisimple Lie group. Referring to a classical decomposition of a Lie algebra of a compact Lie group, associated with root subspaces, we are going to describe its normal subgroups in terms of ideals of the algebra. For an introduction to the root decomposition of Lie algebras of compact Lie groups see [4]. For some specific properties of Lie algebras of holomorphic automorphism groups of bounded circular domains see [2], [3].

By Proposition 4.12 in [1], for any compact Lie group K and its maximal torus (i.e. maximal compact, abelian and connected subgroup in K) T we have

$$L(K) = L(T) + \Sigma_{\theta\in\Theta} V(\theta), \tag{1}$$

where:

— $L(T)$, $L(K)$ denote Lie algebras of T and K respectively;

— Θ is a certain set of group homomorphisms $(T, \circ) \to (\mathbb{R}_{/2\pi\mathbb{Z}}, +)$;

— $\forall\theta \in \Theta$ $V(\theta)$ is a two dimensional real vector space;

— there exists a frame in $V(\theta)$ such that $\forall t \in K$ operator $\mathrm{Ad}(t)_{|V(\theta)}$, defined as a differential of a map $K \ni g \mapsto tgt^{-1}$ at the point $g = e$ (e is the identity of K), has a matrix of the form

$$\begin{bmatrix} \cos(\theta(t)) & -\sin(\theta(t)) \\ \sin(\theta(t)) & \cos(\theta(t)) \end{bmatrix}.$$

For details see [1].

For any $\theta \in \Theta$ a differential $d\theta_e$ is a Lie algebra homomorphism $L(K) \to \mathbb{R}$. For any vector field $X \in L(T)$ operator $\mathrm{ad}(X)_{|V(\theta)}$ defined as a differential of the map $T \ni t \mapsto \mathrm{Ad}(t)_{|V(\theta)}$, has a matrix $d\theta_e(X)\mathbf{J}$, with $\mathbf{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Let Δ be a set of differentials of all functions from Θ. Elements of Δ are called roots.

In the paper we study properties of components of the right hand side of (1) in the case when K is a group of linear automorphisms of a bounded circular domain $D \subset \mathbb{C}^n$.

We assume that $z_1, \ldots, z_n$ be such a fixed system of coordinates in $\mathbb{C}^n$ that all matrices of elements of T are of the form $\mathrm{diag}(e^{i\theta_1(t)}, \ldots, e^{i\theta_n(t)})$, $t \in$T. A proof of the existence of such coordinate system can be found in [2] (see Corollary 12). $\theta_1, \ldots, \theta_n$ are group homomorhisms $(T, \circ) \to (\mathbb{R}_{/2\pi}\mathbb{Z}, +)$. $x_1, \ldots, x_n, y_1, \ldots, y_n$ be a system of real coordinates defined on $\mathbb{C}^n$ by a formula $z_k = x_k + iy_k$, $k = 1, \ldots, n$. Let $I(n) = \{1, \ldots, n\}$.

Denote $\alpha_k = d\theta_k$, $k \in I(n)$. By Theorem 14 of [3], for any $\alpha \in \Delta$ there exist $j, k \in I(n)$ such that $j \neq k$ and $\alpha = \alpha_j - \alpha_k$.

Given any $j \in I(n)$ define: $[j] := \{k \in I(n) : \alpha_j - \alpha_k \in \Delta \text{ or } \alpha_j = \alpha_k\}$, $\Delta[j] := \{\beta \in \Delta : \exists k \in [j] \text{ such that } |\beta| = |\alpha_j - \alpha_k|\}$, $L(T)[j] := [V(\theta), V(\theta)]$, $L(K)[j] := L(T)[j] + \Sigma_{\beta \in \Delta[j]} V(\beta)$. Given any vector space V, denote by $V^{\mathbb{C}}$—its complexification.

We give proofs of the following theorems:

THEOREM 1. *A set* $\mathcal{Z} := \{X \in L(T) : \forall \beta \in \Delta\ \beta(X) = 0\}$ *is a center of* $L(K)$ *and* $1 \leq \dim_{\mathbb{R}} \mathcal{Z} = r - \dim_{\mathbb{R}} \mathrm{lin}(\Delta) \leq r$, *where* $r = \dim_{\mathbb{R}} L(T)$ *and* $\mathrm{lin}(\Delta)$ *is a real linear space spanned by elements of* Δ.

THEOREM 2. *Let* $\beta = d\theta \in \Delta$ *be any fixed root. Given any nonzero* $B \in V(\theta)$ *denote by* $\mathbf{B}$ *a matrix of* B *in coordinates* $x_1, \ldots, x_n, y_1, \ldots, y_n$. *Then:*

a. $\mathbf{B} = \begin{bmatrix} \mathbf{B}' & -\mathbf{B}'' \\ \mathbf{B}'' & \mathbf{B}' \end{bmatrix}$, $\mathbf{B}'^T = -\mathbf{B}'$, $\mathbf{B}''^T = \mathbf{B}''$. *If* b'_{jk}, b''_{jk} $(j, k = 1, \ldots, n)$ *are entries of matrices* $\mathbf{B}'$ *and* $\mathbf{B}''$ *respectively, then* $b'_{jk} = b''_{jk} = 0$ *for all such* (j, k) *that* $|\beta| \neq |\alpha_j - \alpha_k|$.

b. *A vector field defined on* D *by a formula*

$$X = \Sigma_{jk}(b'_{jk} + ib''_{jk})z_j\partial_k + \Sigma_{jk}(b'_{jk} - ib''_{jk})\overline{z}_j\overline{\partial}_k,$$

with b'_{jk}, b''_{jk} *as in point a, generates one-parameter group of transformations of* D *included in* K *with the Lie algebra spanned by* B.

THEOREM 3.

a. *A relation* $\approx$ *in* $I(n)$ *defined by formula:* $j \approx k \Leftrightarrow j \in [k]$, *is an equivalence in* $I(n)$. *Set* $I^*(n) := I(n)_{/\approx}$.

b. $\forall j,k \in I(n)$ $(k \notin [j] \Leftrightarrow \Delta[j] \cap \Delta[k] = \emptyset)$;
c. $L(T) = \Sigma_{j \in I^*(n)} L(T)[j] + \mathcal{Z}$ *is a direct sum of subalgebras.*
d. $L(K) = \Sigma_{j \in I^*(n)} L(K)[j] + \mathcal{Z}$ *is a direct sum of ideals.*

Proof of Theorem 1.

Since every $\beta \in \Delta$ is a linear functional on $L(T)$, a condition $\forall \beta \in \Delta$ $\beta(X) = 0$ defines a linear subspace $\mathcal{Z}$ in $L(T)$, whose dimension obviously satisfies inequality $1 \leq \dim_{\mathbb{R}} \mathcal{Z}$ (D is circular - a vector field generating one-parameter group of rotations $D \ni z \mapsto e^{i\theta} z \in D$ belongs to $\mathcal{Z}$).

By a classical theorem of linear algebra: $\dim_{\mathbb{R}} \mathcal{Z} + \dim_{\mathbb{R}} \operatorname{lin}(\Delta) = \dim_{\mathbb{R}} L(T) = r$, hence $\dim_{\mathbb{R}} \mathcal{Z} = r - \dim_{\mathbb{R}} \operatorname{lin}(\Delta) \leq r$. $\mathcal{Z}$ is an abelian Lie algebra. Since by (1), $\forall Y \in L(T)$ $Y = Y_0 + \Sigma_{\theta \in \Theta} Y_\theta$, where Y_θ is a component of Y from the space $V(\theta), Y_0$ — that from $L(T)$, we have for all $X \in \mathcal{Z}$ $[X,Y] = \operatorname{ad}(X)Y = \operatorname{ad}(X)(Y_0 + \Sigma_{\theta \in \Theta} Y_\theta) = \Sigma_{\theta \in \Theta} d\theta(X) Y_\theta = 0$. $\mathcal{Z}$ is a maximal subspace in $L(T)$ with this property. ■

Proof of Theorem 2:

Let $B \in V(\theta)$ be nonzero. B is an element of the Lie algebra of group of linear transformations of $\mathbb{C}^n$, so it has a natural matrix representation. Denote $\mathbf{B}$—a matrix of B in real coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$.

Ad point a: B is a complex linear map, hence $\mathbf{B}\mathbf{J}_n = \mathbf{J}_n\mathbf{B}$, where $\mathbf{J}_n = \begin{bmatrix} 0 & -1_n \\ 1_n & 0 \end{bmatrix}$, so if $\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 \\ \mathbf{B}_3 & \mathbf{B}_4 \end{bmatrix}$, then $\mathbf{B}\mathbf{J}_n = \begin{bmatrix} \mathbf{B}_1 & -\mathbf{B}_2 \\ \mathbf{B}_3 & -\mathbf{B}_4 \end{bmatrix}$, and $\mathbf{J}_n\mathbf{B} = \begin{bmatrix} -\mathbf{B}_1 & -\mathbf{B}_2 \\ \mathbf{B}_3 & \mathbf{B}_4 \end{bmatrix}$. By comparison of both sides we obtain $\mathbf{B}_3 = -\mathbf{B}_2$, $\mathbf{B}_1 = \mathbf{B}_4$. Thus $\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 \\ \mathbf{B}_2 & \mathbf{B}_1 \end{bmatrix}$. $L(K) \subset L(\mathbf{U}(n))$, hence $\mathbf{B}^T = -\mathbf{B}$ and we conclude that $\mathbf{B}_1^T = -\mathbf{B}_1, \mathbf{B}_2^T = \mathbf{B}_2$. Anticipating a proof of the implication: $|\alpha_j - \alpha_k| = |\beta| \Leftrightarrow b'_{jk} = b''_{jk} = 0$, we prove the point b of the Theorem.

Denote $\mathfrak{f}$ — Lie algebra of vector fields on D generating one-parameter subgroups in K. It can easily be checked that a composition of maps:

$$B \mapsto \begin{bmatrix} \mathbf{B}' & -\mathbf{B}'' \\ \mathbf{B}'' & \mathbf{B}' \end{bmatrix} \mapsto \Sigma_{jk}(b'_{jk} + ib''_{jk}) z_j \frac{\partial}{\partial z_k} + \Sigma_{jk}(b'_{jk} - ib''_{jk}) \overline{z}_j \frac{\partial}{\partial \overline{z}_k}$$

is a Lie algebra homomorphism of $L(K)$ into $\mathfrak{f}$. Actually, it is an isomorphism.

By Lemma 11 of [3] it follows that for any $B \in V(\theta)$ its matrix $\mathbf{B}$ in the standard frame satisfies the following conditions: $\mathbf{B} = \begin{bmatrix} \mathbf{B}' & -\mathbf{B}'' \\ \mathbf{B}'' & \mathbf{B}' \end{bmatrix}$ and $b'_{jk} = b'_{jk} = 0$ for all such (j,k) that $|\alpha_j - \alpha_k| = |\beta|$. ■

Proof of Theorem 3: Ad a: Obviously for any $j,k \in I(n)$ we have: $j \in [j]$ and $(j \in [k] \Leftrightarrow k \in [j])$. Suppose that $j \in [k]$ and $k \in [l]$. Then there

exist $\beta, \gamma \in \Delta$ such that $\beta = \alpha_j - \alpha_k, \gamma = \alpha_k - \alpha_l$. Thus $\beta + \gamma = \alpha_j - \alpha_l$. We have to show that there exists nonzero root subspace in $L(K)^{\mathbb{C}}$ with the root $\beta + \gamma$.

$L(K)$ is isomorphic to the algebra $\mathfrak{f}$ of vector fields on D generating all one-parameter subgroups of K. Given any root $\beta \in \Delta$, denote $\mathfrak{f}^{\beta}$ — a root subspace in $\mathfrak{f}^{\mathbb{C}}$ corresponding to β.

By [2] vector fields $X \in \mathfrak{f}^{\beta}$, $Y \in \mathfrak{f}^{\gamma}$ can locally (and in fact globally) be written in the following form: $X = \Sigma^{*}_{jk} b_{jk} z_j \partial/\partial z_k + \Sigma^{*}_{jk} b'_{jk} \overline{z}_j \partial/\partial \overline{z}_k$, $Y = \Sigma^{**}_{jk} c_{jk} z_j \partial/\partial z_k + \Sigma^{**}_{jk} c'_{jk} \overline{z}_j \partial/\partial \overline{z}_k$, where summation indices in Σ^{*}_{jk} run over all pairs (j,k) such that $\alpha_j - \alpha_k = \beta$ and in Σ^{**}_{jk}—those with $\alpha_j - \alpha_k = \gamma$. Thus $[X,Y] = \Sigma^{\#}_{jk} d_{jk} z_j \partial/\partial z_k + \Sigma^{\#}_{jk} d'_{jk} \overline{z}_j \partial/\partial \overline{z}_k$, where $d_{jk} = \Sigma^{n}_{l=1}(b_{jl}c_{lk} - c_{jl}b_{lk})$, $d'_{jk} = \Sigma^{n}_{l=1}(b'_{jl}c'_{jk} - c'_{jl}b'_{jk})$ and summation indices in $\Sigma^{\#}_{jk}$ run over all pairs (j,k) such that $\alpha_j - \alpha_k = \beta + \gamma$. $[X,Y] \in \mathfrak{f}^{\beta+\gamma}$, so $\beta + \gamma \in \Delta$. This completes the proof of the assertion that $\approx$ is an equivalence relation.

LEMMA. *For any $\theta', \theta'' \in \Theta$*

i. $[V(\theta'), V(\theta'')] = \{0\}$ *when θ', θ'' are not in the same $\Delta[l]$.*

ii. $[V(\theta'), V(\theta'')] \subset \Sigma_{d\theta \in \Delta[l]} V(\theta)$, *when $\theta', \theta'' \in \Delta[l], \theta' \neq \theta''$.*

iii. $[V(\theta'), V(\theta'')] \subset L(T)$, *when $\theta' = \theta''$.*

Proof of Lemma: By the isomorphism of $L(K)$ and $\mathfrak{f}$, there is a one to one correspondence between spaces $V(\theta)$ and spaces $\mathfrak{f}^{d\theta} + \mathfrak{f}^{-d\theta}$ for all $\theta \in \Theta$. All assertions of Lemma can be proved directly by a computation in local coordinates $z_1, \ldots, z_n$ on D. ∎

Ad *b*: For any $\theta \in \Theta$ $[V(\theta), V(\theta)]$ is at most one dimensional subspace in $L(T)$. Hence $L(K)[l] = \Sigma_{d\theta \in \Delta[l]}(V(\theta) + [V(\theta), V(\theta)])$ is a Lie subalgebra in $L(K)$ which is $\mathrm{ad}(L(T))$ — invariant. Since there are no nonzero $\mathrm{ad}(L(T))$—invariant vectors in $V(\theta)$, we conclude that if $j \notin [l]$ then $\Delta[j] \cap \Delta[l] = \emptyset$.

Ad c. We have to show that $\Sigma_{l \in I^*(n)} L(T)[l] + \mathcal{Z} = L(T)$. By the Lemma, $\Sigma_{l \in I^*(n)} L(T)[l]$ is the unique nonzero subalgebra of $[L(K), L(K)]$ which has nonzero intersection with $L(T)$. Since K is a compact Lie group, by complete reducibility of linear space $L(K)$ with respect to family $\mathrm{ad}(L(K))$ of operators in $L(K)$ (see: [1], Theorem 3.20) we have $L(K) = [L(K), L(K)] + \mathcal{Z}$ and this completes the proof of the assertion.

Ad d. $L(K) = \Sigma_{l \in I^*(n)} L(K)[l] + \mathcal{Z}$ is a direct sum of subalgebras. By the Lemma, $[L(K), L(K)[l]] \subset L(K)[l]$ i.e. $L(K)[l]$ is an ideal in $L(K)$. ∎

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$\mathbb{Y}$-VALUED SOLUTIONS FOR SEMILINEAR GENERALIZED WAVE EQUATION

1. Introduction

Let R^n be n-dimensional Euclidean space, $n \in \mathbb{N}$ and $\mathbb{N}$ the set of integers.

ASSUMPTION 1. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a bounded domain with boundary Γ of class C^{2m}, $m \in \mathbb{N}$ and closure $\overline{\Omega}$. Let $t \in \langle 0, T\rangle$, $T \in \mathbb{R}^+ = \langle 0, \infty)$ and let functions $\phi, \psi : \Omega \to \mathbb{R}$ and $a_{pq} : \overline{\Omega} \times \langle 0, T\rangle \to \mathbb{R}$ be given, p, q are multiindices

$$p = (p_1, p_2, \ldots, p_m), \quad q = (q_1, q_2, \ldots, q_m), \quad |p| = \sum_{i=1}^{m} p_i, |q| = \sum_{i=1}^{m} q_i.$$

Let $H^s(\Omega)$ and $H_0^s(\Omega)$ be Sobolev spaces with the norms $\|.\|_{H^s}, s \in \mathbb{R}$. Let $A_0(t, x, D)$, $t \in \langle 0, t\rangle$, be a family linear elliptic operators [5] of order $2m, m \in \mathbb{N}$, in the divergence form

$$A_0(t, x, D) = \sum_{|p|=|q|=0}^{m} (-1)^{|p|} D^p (a_{pq}(x, t) D^q). \tag{1}$$

ASSUMPTION 2. $a_{pq} \in C^{2m,1}(\overline{\Omega} \times \langle 0, T\rangle)$, $a_{pq} = a_{qp}$, $|p| \leq m$, $|q| \leq m$.

ASSUMPTION 3. The operators $A_0(t, x, D)$, $t \in \langle 0, T\rangle$, are uniformly strongly elliptic in Ω, i.e. there is a constant $c > 0$ that

$$\sum_{|p|=|q|=0}^{m} (-1)^{|p|} a_{pq}(x, t) \xi^p \xi^q \geq c|\xi|^{2m},$$

for every $x \in \overline{\Omega}$, $t \in \langle 0, T\rangle$, $\xi \in \mathbb{R}^n$.

Then the operators $A_0(t, x, D)$ for $t \in \langle 0, T\rangle$ satisfy Gårding's inequality, i.e. there exist constants $C_1 \geq 0$ such that

$$a(v,v,t) \geq C_1\|v\|^2_{H^m(\Omega)} - C_2\|v\|^2_{L_2(\Omega)}$$

for any $v \in H_0^m(\Omega)$ and $t \in \langle 0,T\rangle$, where the bilinear form $a(v,w,t)$ being given by the formula

$$(2) \quad a(v,w,t) = \sum_{|p|=|q|=0}^{m} \int_{\Omega} a_{pq}(x,t)D^p v(x) D^q w(x)dx, \qquad w,v \in H^m(\Omega).$$

If $C_2 \neq 0$ we can replace the operators $A_0(t,x,D)$ by the operators $A(t,x,D) = A_0(t,x,D) + \lambda\mathbb{I}$, where $\mathbb{I}$ — the identity operator, and $\lambda \geq C_2$. Then for any $v \in W_0^m(\Omega)$ and $t \in \langle 0,T\rangle$

$$(3) \qquad a(v,v,t) \geq C_1\|v\|^2_{H^m(\Omega)}.$$

In this paper existence of solutions of the following equation

$$(\mathrm{E}) \quad u_{tt}+A(t,x,D)u = f(t,x,u,u_t,Du,\ldots,D^{m-1}u), \quad x \in \Omega, \quad t \in \langle 0,T\rangle,$$

with the initial conditions

$$(\mathrm{IC}) \qquad u(x,0) = \phi(x), \quad u_t(x) = \psi(x), \quad x \in \Omega,$$

and the boundary conditions

$$(\mathrm{BC}) \qquad D^\beta u|_\Gamma = 0, \quad \text{for } |\beta| \leq m-1,\ t \in \langle 0,T\rangle,\ \beta = (\beta_1,\ldots,\beta_n)$$

has been investigated.

Now the problem (E), (IC), (BC) will be set in an abstract form and the theory of an evolution system $U(t,s)$, $0 \leq s \leq t \leq T$, on a certain Banach space will be applied in order to find solutions. This provides us with the existence and uniqueness of solutions in the sense of this Banach space [4], [5].

With the elliptic operators $A(t,x,D)$, $t \in \langle 0,T\rangle$, we associate linear operators $A(t)$, $t \in \langle 0,T\rangle$, in $L^2(\Omega)$. This is done as follows:

$$D(A(t)) = D = H^{2m}(\Omega) \cap H_0^m(\Omega) \quad \text{and} \quad A(t)u = A(t,x,D)u \quad \text{for } u \in D.$$

It is obvious that D is dense in $L^2(\Omega)$.

LEMMA 1. *By Assumptions* 1–3 *and density of D in $L^2(\Omega)$ we have:*

(i) *the operator $A(t)$, for every $t \in \langle 0,T\rangle$, can be extended to self-adjoint operator* (*proof* [6], *p.* 126);

(ii) *for any $\alpha \in (0,1)$, the operator $A^\alpha(t)$ is self-adjoint and $D_\alpha = D(A^\alpha(t))$ is also independent of t* (*proof* [5], *p.* 109); *for $\alpha = 1/2$, in our case $D_{1/2} = D(A^{1/2}(t)) = H_0^m(\Omega)$;*

(iii) $a(t,v,w) = (A^{1/2}(t)v, A^{1/2}(t)w)$ (*proof* [5], *p.* 29).

We can set the problem (E), (IC), (BC) as the abstract initial value problem

$$u_{tt} + A(t)u = f_1(t, u, u_t), \tag{4}$$

$$u(0) = \phi, u_t(0) = \psi, \tag{5}$$

where $f_1(t, u, u_t)(x) = f(t, x, u, u_t, Du, \ldots, D^{m-1}u)$.

Next, the problem (4), (5) can be written in the form

$$\frac{dw}{dt} = \mathbb{A}(t)w + F(t, w), \tag{6}$$

$$w_0 = w(0) = \begin{pmatrix} \phi \\ \psi \end{pmatrix}, \tag{7}$$

where $\mathbb{A} = \begin{pmatrix} 0 & \mathbb{I} \\ -A(t) & 0 \end{pmatrix}$, $F(t, w) = \begin{pmatrix} 0 \\ f_1(t, w) \end{pmatrix}$, $w = \begin{pmatrix} u \\ u_t \end{pmatrix}$.

Let $\mathbb{H}_0 = H_0^m(\Omega) \times L_2(\Omega)$ and $D(\mathbb{A}(t)) = \mathbb{D} := (H^{2m}(\Omega) \cap H_0^m(\Omega)) \times H_0^m(\Omega)$, for $t \in \langle 0, T \rangle$. The space $\mathbb{H}_0$ is the Hilbert space with scalar product

$$(w_1, w_2)_{\mathbb{H}_0} = (A^{1/2}(t)v_1, A^{1/2}(t)v_2) + (z_1, z_2),$$

of $w_1 = \begin{pmatrix} v_1 \\ z_1 \end{pmatrix}$, $w_2 = \begin{pmatrix} v_2 \\ z_2 \end{pmatrix}$.

2. An evolution system

Let X be a Banach space with the norm $\|.\|$. For every $t \in \langle 0, T \rangle$, let $A(t) : D(A(t)) \subset X \to X$ be a linear operator in X and $f : \langle 0, T \rangle \times X \to X$ be a function.

DEFINITION 1. A two parameter family $\{U(t,s)\}$ of bounded linear operators $U(t,s)$, $0 \leq s \leq t \leq T$, on X, is called an evolution system if the following two conditions are satisfied:

(i) $U(s,s) = \mathbb{I}$, $U(t,r)U(r,s) = U(t,s)$ for $0 \leq s \leq r \leq t \leq T$,

(ii) $(t,s) \to U(t,s)$ is strongly continuous for $0 \leq s \leq t \leq T$.

DEFINITION 2. A family $\{A(t)\}$, $t \in \langle 0, T \rangle$, of infinitesimal generators of C_0 semigroups on X is called stable if there are constants $M \geq 1$ and $\omega \in R$ such that resolvent set satisfies conditions:

$$\varrho(A(t)) \supset (0, \infty) \quad \text{for } t \in \langle 0, T \rangle$$

and

$$\Big\| \bigcap_{j=1}^{k} R(\lambda; A(t_j)) \Big\| \leq M(\lambda - \omega)^{-k}$$

for $\lambda > \omega$ and for every finite sequence $0 \leq t_1 \leq t_2 \leq \ldots \leq t_k \leq T$, $k = 1, 2, \ldots$; M and ω are called the stability constants.

Remark 1. Any family $\{A(t)\}$, $t \in \langle 0,T\rangle$, of infinitesimal generators of C_0 semigroups of contractions is stable ([4], p. 131).

DEFINITION 3. Let X and Y be Banach spaces with norms $\|.\|$ and $\|.\|_Y$, respectively. Y is densely and continuously imbedded in X, if Y is dense subspace of X and there is a constant $C > 0$ such that $\|w\| \le C\|w\|_Y$ for $w \in Y$.

DEFINITION 4. Let X be a Banach space and $S : D(S) \subset X \to X$ a linear operator in X. The subspace Y of X is an invariant subspace of S if $S : D(S) \cap Y \to Y$.

DEFINITION 5. Let A be a infinitesimal generator of C_0 semigroup $S(s)$, $s \in \mathbb{R}^+$. A subspace Y of X is called A-admissible, if it is an invariant subspace of $S(s)$, $s \in \mathbb{R}^+$, and the restriction of $S(s)$ to Y is a C_0 semigroup in Y (i.e., it is strongly continuous in the norm $\|.\|_Y$).

LEMMA 2. *Let, for each* $t \in \langle 0,T\rangle$, $A(t)$ *be the infinitesimal generator of* C_0 *semigroup* $S_t(s)$, $s \in \mathbb{R}^+$, *on* X. *The following conditions* (H1)–(H3) (*usually referred to as the "hyperbolic" case*):

(H1) $\{A(t)\}$, $t \in \langle 0,T\rangle$, *is a stable family with stability constants* M, ω,

(H2) Y *is* $A(t)$-*admissible, for* $t \in \langle 0,T\rangle$, *and the family* $\{\widetilde{A}(t)\}$, $t \in \langle 0,T\rangle$, *of the parts of* $A(t)$ *in* Y, *is a stable family in* Y *with stability constants* $\widetilde{M}, \widetilde{\omega}$,

(H3) *for* $t \in \langle 0,T\rangle$, $Y \subset D(A(t))$, $A(t)$ *is a bounded operator from* Y *into* X *and* $t \to A(t)$ *is continuous in* $B(Y,X)$ *norm* $\|.\|_{Y\to X}$; *guarantee existence of a unique evolution system* $U(t,s)$, $0 \le s \le t \le T$, *in* X *satisfying*

(E1) $$\|U(t,s)\| \le M\exp[\omega(t-s)] \quad \text{for } 0 \le s \le t \le T,$$

(E2) $$\frac{\partial^+}{\partial t}U(t,s)v|_{t=s} = A(s)v \quad \text{for } v \in Y,\ 0 \le s \le T,$$

(E3) $$\frac{\partial}{\partial s}U(t,s)v = -U(t,s)A(s)v \quad \text{for } v \in Y,\ 0 \le s \le t \le T,$$

where the right-hand derivative in (E2) *and the derivative in* (E3) *are in the strong sense in* X (*proof* [4], *p.* 135).

LEMMA 3. *Let* $\{A(t)\}$, $t \in \langle 0,T\rangle$, *be a stable family of infinitesimal generators of* C_0 *semigroup on* X. *If* $D(A(t)) = D$ *is independent of* t *and* $A(t)v$ *is continuously differentiable in* X *for* $v \in D$, *then there exists a unique evolution system* $U(t,s)$, $0 \le s \le t \le T$, *satisfying* (E1)–(E3) *and*

(E4) $U(t,s)Y \subset Y$ *for* $0 \le s \le t \le T$,

(E5) *$U(t,s)v$ is continuous in Y for $0 \le s \le t \le T$ and $v \in Y$, where Y is D equipped with the norm $\|v\|_Y = \|v\| + \|A(0)v\|$, for $v \in Y = D$. (proof [4], p. 145, see also [1], [3]).*

THEOREM 1. *If Assumptions 1–3 are satisfied, then the family $\{\mathbb{A}(t)\}$ of operators $\mathbb{A}(t)$, $t \in \langle 0,T\rangle$, is generator of a unique evolution system on the space $\mathbb{H}_0$ having the properties* (E1)–(E5).

Proof. At first we will prove that $\mathbb{A}(t)$ is a dissipative operator for each $t \in \langle 0,T\rangle$. By Lemma 1, we have

$$[\mathbb{A}(t)w,w]_{\mathbb{H}_0} = \left[\begin{pmatrix} 0 & \mathbb{I} \\ -A(t) & 0\end{pmatrix}\begin{pmatrix} v \\ z\end{pmatrix}, \begin{pmatrix} v \\ z\end{pmatrix}\right]_{\mathbb{H}_0} = \left[\begin{pmatrix} z \\ -A(t)v\end{pmatrix}, \begin{pmatrix} v \\ z\end{pmatrix}\right]_{\mathbb{H}_0} =$$
$$= (A^{1/2}(t)z, A^{1/2}(t)v) + (-A(t)v,z) = (z, A(t)v) - (A(t)v,z) =$$
$$= (A(t)v,z) - (A(t)v,z) = 0.$$

Thus $\mathbb{A}(t)$ is dissipative for each $t \in \langle 0,T\rangle$.

Now we will prove that for any $\lambda > 0$ the range of $(\lambda E - \mathbb{A}(t))$, for each $t \in \langle 0,T\rangle$ is all of $\mathbb{H}_0$, $E = \begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix}$. We have to prove that equation $(\lambda E - \mathbb{A}(t))w = F$, $F = \begin{bmatrix} h \\ g\end{bmatrix}$, $w = \begin{bmatrix} v \\ z\end{bmatrix}$ has a solution $w \in (H^{2m}(\Omega) \cap H_0^m(\Omega)) \times H_0^m(\Omega)$ for any $(h,g) \in H_0^m(\Omega) \times L^2(\Omega)$. This equation is equivalent to the system of equations $(A(t) + \lambda^2\mathbb{I})v = g + \lambda h$, $z = \lambda v - h$. It is clear that $g + \lambda h \in L^2(\Omega)$. Due to (3) the bilinear form (2) is coercive for $t \in \langle 0,T\rangle$ with $C_2 = 0$. We may therefore apply the Lax–Milgram theorem and derive the existence of a unique weak solution $v \in H_0^m(\Omega)$ of boundary value problem $(A(t) + \lambda^2\mathbb{I})v = g + \lambda h$ for $\lambda > 0$ ([2], p. 43). The coefficients $a_{pq}(x,t)$ of $A(t,x,D)$ (Assumption 2) and the boundary Γ (Assumption 1) are smooth enough that we can apply regularization theory ([2], p. 67). So we obtain $v \in H^{2m}(\Omega)$ and finally $v \in H^{2m}(\Omega) \cap H_0^m(\Omega)$. From the equation $z = \lambda v - h$ and the condition $h \in H_0^m(\Omega)$ we have $z \in H_0^m(\Omega)$. This means that $\mathbb{A}(t)$ is maximal operator for each $t \in \langle 0,T\rangle$. It is also clear that $D(\mathbb{A}(t)) = (H^{2m}(\Omega) \cap H_0^m(\Omega)) \times H_0^m(\Omega)$ is dense in $\mathbb{H}_0$.

All assumptions of Lummer–Phillips Theorem ([4], p. 14) are satisfied, so $\{\mathbb{A}(t)\}$, $t \in \langle 0,T\rangle$, is a family of the infinitesimal generators of C_0 semigroups of contractions on $\mathbb{H}_0$, i.e.,

$$\|T_t(s)\| \le 1 \quad \text{for } s \in \mathbb{R}^+ \quad \text{and} \quad t \in \langle 0,T\rangle.$$

By Remark 1, the family $\{\mathbb{A}(t)\}$, $t \in \langle 0,T\rangle$, is stable with stability constants $M = 1$ and $\omega = 0$. It is obvious that $D(\mathbb{A}(t))$ are independent of $t \in \langle 0,T\rangle$. Assumption 2 implies that $\mathbb{A}(t)w$ is continuously differentiable in $\mathbb{H}_0$ for

$w \in \mathbb{D}$. All assumptions of Lemma 3 are satisfied, then there exists a unique evolution system $U(t,s)$, $0 \le s \le t \le T$, on $\mathbb{H}_0$ satisfying (E1)–(E5).

3. Existence of solutions

At first we recall some definitions.

DEFINITION 1. A function $w : \langle 0,T\rangle \to \mathbb{H}_0$ is said to be a mild solution of the problem (6), (7) if $w \in C(\langle 0,T\rangle, \mathbb{H}_0)$ for any $w_0 \in \mathbb{H}_0$, and w satisfies the following integral equation

$$w(t) = U(t,0)w_0 + \int_0^t U(t,s)F(s,w(s))\,ds, \qquad 0 \le t \le T.$$

DEFINITION 2. A function $w : \langle 0,T\rangle \to \mathbb{H}_0$ is said to be a $\mathbb{Y}$-valued solution of the problem (6), (7), if $w \in C(\langle 0,T\rangle; \mathbb{Y}) \cap C^1((0,T>; \mathbb{H}_0)$ and equation (6) is satisfied in $\mathbb{H}_0$.

The set $\mathbb{Y}$ is domain $D(\mathbb{A}(t)) = \mathbb{D} = (H^{2m}(\Omega) \cap H_0^m(\Omega)) \times H_0^m(\Omega)$ with the norm $\|w\|_{\mathbb{Y}} = \|w\|_{\mathbb{H}_0} + \|\mathbb{A}(0)\|_{\mathbb{H}_0}$ for $w \in \mathbb{D}$.

DEFINITION 3. A function $F : \langle 0,T\rangle \times \mathbb{H}_0 \to \mathbb{H}_0$ is said to be Lipschitz continuous in w, uniformly in $t \in \langle 0,T\rangle$, with constant $L > 0$, if $\|F(t,w_2) - F(t,w_1)\|_{\mathbb{H}_0} \le L\|w_2 - w_1\|_{\mathbb{H}_0}$ for every $t \in \langle 0,T\rangle$, $w_1, w_2 \in \mathbb{H}_0$.

DEFINITION 4. A function $F : \langle 0,\infty) \times \mathbb{H}_0 \to \mathbb{H}_0$ is said to be locally Lipschitz continuous in w, uniformly in t on bounded intervals, if for every constants $r \ge 0$, $\tau \ge 0$ there exists a constant $L(r,\tau)$ such that

$$\|F(t,w_2) - F(t,w_1)\|_{\mathbb{H}_0} \le L(r,\tau)\|w_2 - w_1\|_{\mathbb{H}_0}$$

for every $t \in \langle 0,\tau\rangle$ and $w_1, w_2 \in \mathbb{H}_0$ with $\|w_1\|_{\mathbb{H}_0} \le r$, $\|w_2\|_{\mathbb{H}_0} \le r$.

THEOREM 2. *If Assumptions* 1–3 *are satisfied and the function* $F : \langle 0,T\rangle \times \mathbb{H}_0 \to \mathbb{H}_0$ *is continuous in* $t \in \langle 0,T\rangle$ *and Lipschitz continuous in* w*, uniformly in* $t \in \langle 0,T\rangle$*, then for every* $w_0 \in \mathbb{H}_0 = H_0^m(\Omega) \times L_2(\Omega)$ *there exists a unique mild solution* $w \in C(\langle 0,T\rangle; H_0^m(\Omega) \times L_2(\Omega))$ *of the problem* (6), (7).

It has been proved in general case in [4], [5].

THEOREM 3. *If Assumptions* 1–3 *are satisfied and function* $F : \langle 0,\infty) \times \mathbb{H}_0 \to \mathbb{H}_0$ *is continuous in* t *for* $t \ge 0$ *and locally Lipschitz continuous in* w*, uniformly in* t *on bounded intervals, then for every* $w_0 \in \mathbb{H}_0$ *there exists a unique mild solution* $w \in C(\langle 0,t_{\max}), H_0^m(\Omega) \times L_2(\Omega))$ *of the problem* (6), (7) *with either* $t_{\max} = \infty$ *or* $t_{\max} < \infty$*. Moreover, if* $t_{\max} < \infty$ *then* $\lim_{t\to t_{\max}} \|w(t)\|_{\mathbb{H}_0} = \infty$.

Proof. The proof is similar to that of Theorem 1.4 ([4], p. 185), but in our case we have to put $M_2(t_0) = \max\{\|U(t,s)\|;\ 0 \le s \le t \le t_0 + 1\}$ and use integral equation

$$w(t) = U(t,t_0)w_0 + \int_{t_0}^{t} U(t,s)F(s,w(s))\,ds.$$

THEOREM 4. *If Assumptions 1–3 are satisfied and function* $F : \langle 0,T\rangle \times \mathbb{Y} \to \mathbb{Y}$ *is Lipschitz continuous in* $\mathbb{Y}$*, uniformly in* $t \in \langle 0,T\rangle$ *and for each* $w \in \mathbb{Y}$ *continuous from* $\langle 0,T\rangle$ *into* $\mathbb{Y}$*, then for* $w_0 \in \mathbb{Y}$ *the problem* (6), (7) *has a unique* $\mathbb{Y}$*-values solution on* $\langle 0,T\rangle$*, i.e.*

$$w \in C(\langle 0,T\rangle; (H^{2m}(\Omega) \cap H_0^m(\Omega)) \times H_0^m(\Omega)) \cap C^1((0,T>; H_0^m(\Omega) \times L_2(\Omega)).$$

Proof. In a standard way we can prove existence of the mild solution $\omega \in C(\langle 0,T\rangle; \mathbb{Y})$ which satisfies the integral equation

$$\omega(t) = U(t,0)w_0 + \int_0^t U(t,s)F(s,\omega(s))\,ds$$

in $\mathbb{Y}$ and a fortiori in $\mathbb{H}_0$ for a given $w_0 \in \mathbb{Y}$.

Let $g(s) = F(s,\omega(s))$, $s \in \langle 0,T\rangle$. Then, by the assumptions of our theorem, it follows that $g(s) \in \mathbb{Y}$ for $s \in \langle 0,T\rangle$ and $g \in C(\langle 0,T\rangle;\mathbb{Y})$. Theorem 5.2 ([4], p. 146) guarantees existence of a unique $\mathbb{Y}$-valued solution w on $\langle 0,T\rangle$ for the linear problem

$$\begin{cases} \dfrac{dw}{dt} + \mathbb{A}(t)w = g(t), \\ w(0) = w_0 \end{cases} \tag{8}$$

for $g \in C(\langle 0,T\rangle;\mathbb{Y})$ and $w_0 \in \mathbb{Y}$. This solution is then clearly also a mild solution of (8) and therefore

$$w(t) = U(t,0)w_0 + \int_0^t U(t,s)g(s)\,ds =$$

$$= U(t,0)w_0 + \int_0^t U(t,s)F(s,\omega(s))\,ds = \omega(t).$$

So $w = \omega$ and ω is a $\mathbb{Y}$-valued solution of problem (6), (7) on $\langle 0,T\rangle$.

THEOREM 5. *If Assumptions 1–3 are satisfied and the function* $F : \langle 0,T\rangle \times \mathbb{Y} \to \mathbb{Y}$ *is continuous in* $t, T \in \mathbb{R}^+$*, and locally Lipschitz continuous in* $\mathbb{Y}$*, uniformly in* t *on* $\langle 0,T\rangle$*, then for every* $w_0 \in \mathbb{Y}$ *the problem* (6), (7) *has a unique* $\mathbb{Y}$*-valued solution*

$$w \in C(\langle 0, t_{\max}); (H^{2m}(\Omega) \cap \\ \cap H_0^m(\Omega)) \times H_0^m(\Omega)) \cap C^1((0, t_{\max}); H_0^m(\Omega) \times L^2(\Omega))$$

on a maximal interval $\langle 0, t_{\max})$ $t_{\max} \leq T$. *Moreover, if* $t_{\max} < T$, *then*

$$\lim_{t \to t_{\max}} [\|w(t)\|_{\mathbb{H}_0} + \|\mathbb{A}(0)w(t)\|_{\mathbb{H}_0}] = \infty.$$

Proof. The proof of this theorem is similar to that of Theorem 4. It has been also used the results of Theorem 3.

Remark 2. Similar results, as in Theorems 2–4, have been obtained in the papers [1], [3], under a little bit weaker assumptions.

Due to Yamaguchi's Theorem ([7], Appendix) the following Lemma holds.

LEMMA 4. *Let* $f(t, x, a)$ *be defined on* $\langle 0, T\rangle \times \overline{\Omega} \times R^{N+1}$, $a = (a_0, a_1, \ldots \ldots, a_N) \in R^{N+1}$, $N := \frac{(n+m-1)!}{n!(m-1)!}$. *Let* $f(x, t, a)$ *be of* C^{sm+1}*-class in* $(x, a) \in \overline{\Omega} \times R^{N+1}$ *and* $D_x^k D_a^l f(x, t, a)$, $0 \leq k + l \leq sm + 1$, *be continuous in* t *on* $t \in \langle 0, T\rangle$, $s \in \mathbb{N}$ *and* $s > [\frac{n-2}{2m}] + 1$. *Let* $B(Q) = \{a : a \in R^{N+1}; |a_i| \leq Q, i = 0, 1, 2, \ldots, N\}$, *where* Q *is some positive real number. Set*

$$h_Q(t) = \max_{0 \leq k+l \leq sm+1} \sup_{x \in \overline{\Omega}\, a \in B(Q)} |D_x^k D_a^l f(t, x, a)| \quad \text{and denote}$$

$$H_Q = \{u \in C(\langle 0, T\rangle, H^{sm}(\Omega)) \cap C^1(\langle 0, T\rangle, H^{(s-1)m}), \\ |D^\beta u(t, x)| \leq Q, \quad |\beta| \leq m - 1, \quad |u_t(t, x)| \leq Q\}.$$

Then the following assertions hold:

(L1) *there exists a positive constant* C_1 *and a function* $h_Q : \langle 0, T\rangle \to \langle 0, T\rangle$ *defined above such that for any* $u \in H_Q$ *we have*

$$\|f(t, \ldots, u(t, .), u_t(t, .), Du(t, .), \ldots, D^{m-1}u(t, .))\|_{\mathbb{H}^{s-1}} \leq \\ \leq C_1 h_Q(t)[\|u(t, .)\|_{H^{sm}} + \|u_t(t, .)\|_{H^{(s-1)m}} + 1],$$

(L2) *there exists a positive constant* C_2 *such that for any* $u(t, .), \widetilde{u}(t, .) \in H_Q$ *satisfying the conditions*

$$[\|u(t, .)\|_{H^{sm}} + \|u_t(t, .)\|_{H^{(s-1)m}}] \leq C_3,$$
$$[\|\widetilde{u}(t, .)\|_{H^{sm}} + \|\widetilde{u}_t(t, .)\|_{H^{(s-1)m}}] \leq C_3,$$

for some positive constant C_3, *we have*

$$\|f(t, ., \widetilde{u}(t, .), \widetilde{u}_t(t, .), D\widetilde{u}(t, .), \ldots, D^m\widetilde{u}(t, .)) +$$

$$- f(t,.,u(t,.),u_t(t,.),Du(t,.),\ldots,D^{m-1}u(t,.))\|_{\mathbb{H}^{s-1}} \leq$$

$$\leq C_2 h_Q(t)[\|\widetilde{u}(t,.)-u(t,.)\|_{H^{sm}} + \|\widetilde{u}_t(t,.)-u_t(t,.)\|_{H^{(s-1)m}}].$$

Remark 3. If we denote

$$F = \begin{pmatrix} 0 \\ f \end{pmatrix}, \quad w(t) = \begin{pmatrix} u(t,.) \\ u_t(t,.) \end{pmatrix}, \quad \widetilde{w}(t) = \begin{pmatrix} \widetilde{u}(t,.) \\ \widetilde{u}_t(t,.) \end{pmatrix}$$

we obtain from Lemma 4 the following conditions

$(\widetilde{\mathrm{L}}1)$ $\quad \|F(t,w(t,.))\|_{\mathbb{H}_s} = \|f\|_{H^{(s-1)m}} \leq C_1 h_Q(t)[\|w(t,.)\|_{\mathbb{H}_s} + 1],$

$(\widetilde{\mathrm{L}}2)$ $\quad \|F(t,\widetilde{w}(t,.)) - F(t,w(t,.))\|_{\mathbb{H}_s} \leq C_2 h_Q(t)\|\widetilde{w}(t,.)-w(t,.)\|_{\mathbb{H}_s},$

where $\mathbb{H}_s = H^{sm}(\Omega) \times H^{(s-1)m}(\Omega)$.

The condition $(\widetilde{\mathrm{L}}1)$ means that, if $w(t,.) \in \mathbb{H}_s$ then $F(t,w(t,.)) \in \mathbb{H}_s$ for $t \in \langle 0,T\rangle$. The condition $(\widetilde{\mathrm{L}}2)$ means that the function F is locally Lipschitz continuous in w with respect to norm $\mathbb{H}_s$.

THEOREM 6. *Let*

(i) $f \in C^{2m+1}$, *and a function f satisfies assumptions of Lemma* 4 *with* $s = 2$,

(ii) *for any* $x \in \partial\Omega$, $t \in \langle 0,T\rangle$ $D_x^k D_a^l f(x,t,0) = 0$ *for* $k + l \leq m-1$,

(iii) *Assumptions* 1–3 *are satisfied*,

then for $(\phi,\psi) \in (H^{2m}(\Omega) \cap H_0^m(\Omega)) \times H_0^m(\Omega)$ *there exists a unique local $\mathbb{Y}$-valued solution of the problem* (E), (IC), (BC), *i.e.*

$$u \in C(\langle 0,t_{\max}); H^{2m}(\Omega)\cap$$
$$\cap H_0^m(\Omega)) \cap C^1(\langle 0,t_{\max}), H_0^m(\Omega)) \cap C^2((0,t_{\max}), L_2(\Omega))).$$

If $t_{\max} < T$, *then*

$$\lim_{t\to t_{\max}} [\|u(t,.)\|_{H_0^m(\Omega}+$$
$$+\|A(0)u(t,.)\|_{L_2} + \|u_t(t,.)\|_{H_0^m(\Omega)} + \|u_t(t,.)\|_{L_2(\Omega)}] = \infty.$$

Proof. The norm $\|F(t,w)\|_{\mathbb{Y}}$ is equivalent to $\|F(t,w)\|_{H_2}$. Lemma 4 with $s = 2$ $(1 \leq n < 2m+2)$ and condition (ii) guarantee that $F : \langle 0,T\rangle \times \mathbb{Y} \to \mathbb{Y}$ and it is locally Lipschitz continuous in $\mathbb{Y}$, uniformly in t on $\langle 0,T\rangle$. So all assertions of Theorem 5 are satisfied and this implies the thesis of our theorem.

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A CHARACTERIZATION OF LOW DEGREE POLYNOMIALS

This paper concerns with the solution of a functional equation that characterizes low degree polynomials. The method used for solving the equation is simple and elementary. Here, an answer is also provided to a problem posed by Walter Rudin in the *American Mathematical Monthly* [11] in a general setting.

1. Introduction

Let R be the set of all real numbers. A function $A : R \to R$ is said to be an *additive function* on reals if

$$A(x+y) = A(x) + A(y)$$

for all real numbers x and y. There are many papers dealing with the various aspects of additive functions. A comprehensive review on additive functions can be found in [9].

It is well known that for quadratic polynomials the Mean Value Theorem takes the form

$$\frac{f(x)-f(y)}{x-y} = f'\left(\frac{x+y}{2}\right). \tag{1}$$

It was shown in [1] (and also [6]) that the solution of the functional differential equation (1) is of the form $f(x) = ax^2 + bx + c$, where a, b and c are arbutrary real constants. This has the following interpretation. Let $f(t)$ denote the position of a moving object at time t. If the mean velocity

$$\frac{f(y)-f(x)}{y-x}$$

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during every interval $[x, y]$ is equal to the velocity $f'(\frac{x+y}{2})$ at the arithmetic mean $\frac{x+y}{2}$ of the end points x and y of the interval $[x, y]$, then the trajectory of the object is a parabola or a line. The above functional equation (1) is a special case of the following equation

$$\frac{f(x) - f(y)}{x - y} = \phi(\eta(x, y)),$$

where $\eta(x, y)$ is an *apriori* known function of x and y. Note that this equation contains no derivative and no mean value. In a recent paper, Aczel and Kuczma [2] have determined the solution of the above functional equation assuming $\eta(x, y)$ to be either arithmetic, or harmonic or geometric mean of x and y. Several authors (see [1], [2], [6], [8]) have treated equations of this type. If $\eta(x, y) = x + y$, then the above equation characterizes quadratic polynomials. For a generalization of (1) to higher derivatives the reader may refer to [3], [7] and [8]. For a generalization of (1) that characterizes polynomials of degree at most n the reader may refer to [3], [4], and [7]. The motivation behind the study of the above functional equations for characterizing polynomials can be found in [1], [2], [3], [5], [6], [7], [8], [10], [12] and references therein.

In the *American Mathematical Monthly* [11] the following problem was proposed by Walter Rudin: *"Let s and t be given real numbers. Find all differentiable functions f on the real line which satisfy*

$$\text{(RE)} \qquad f'(sx + ty) = \frac{f(y) - f(x)}{y - x}$$

for all real x,y, with $x \neq y$." Note that any solution of (RE) is intrinsically differentiable. In fact, if f is a solution of (RE), then $f \in C^\infty(R)$. The equation given by Rudin is a special case of the following functional equation

$$\text{(2)} \qquad g(sx + ty) = \frac{f(y) - f(x)}{y - x}$$

for all $x, y \in R$ with $x \neq y$. The solution of (2) was given by Baker [13] in the following theorem. *If s,t are given real numbers, then the necessary and sufficient condition for f,g to satisfy $f(y) - f(x) = (y - x)g(sx + ty)$ for all x,y is as follows: If $s = t = 0$ or $s^2 \neq t^2$, then f is a polynomial of degree at most 1 and $g = f'$. If $s = t \neq 0$, then f is a polynomial of degree at most 2 and $g(x) = f'(x/2t)$. If $s = -t \neq 0$, then $f(x) = a + A(x)$ and $g(x) = A(x/t)/(x/t)$ for $x \neq 0$, where a is a real constant and A is an additive function.* Also, the solution of (2) (and also of (RE)) was obtained independently by the last two authors of this paper.

It is the purpose of this paper to present an elementary and simple technique for determining solution of the following functional equation:

$$\text{(FE)} \qquad \frac{f(x)-g(y)}{x-y} = h(sx+ty)$$

for all real x, y with $x \neq y$. Here s and t are *a priori* known real parameters. This equation generalizes (2) and characterizes polynomials of low degrees.

2. The solution of the functional equation

Now we proceed to find the general solution of (FE) with no regularity assumptions (differentiability, continuity, measurability, etc.) imposed on h, g and f.

THEOREM 1. *Let s and t be the real parameters. Functions $f, g, h : R \to R$ satisfy (FE) for all $x, y \in R, x \neq y$ if and only if*

$$f(x) = \begin{cases} ax+b & \text{if } s = 0 = t \\ ax+b & \text{if } s = 0,\ t \neq 0 \\ \alpha t x^2 + ax + b & \text{if } s = t \neq 0 \\ \dfrac{A(tx)}{t} + b, & \text{if } s = -t \neq 0 \\ \beta x + b & \text{if } s^2 \neq t^2 \end{cases}$$

$$g(y) = \begin{cases} ay+b & \text{if } s = 0 = t \\ ay+b & \text{if } s = 0,\ t \neq 0 \\ \alpha t y^2 + ay + b & \text{if } s = t \neq 0 \\ \dfrac{A(ty)}{t} + c, & \text{if } s = -t \neq 0 \\ \beta y + b & \text{if } s^2 \neq t^2 \end{cases}$$

$$h(y) = \begin{cases} \text{arbitrary with } h(0) = a & \text{if } s = 0 = t \\ a & \text{if } s = 0,\ t \neq 0 \\ \alpha y + a & \text{if } s = t \neq 0 \\ \dfrac{A(y)}{y} + \dfrac{(c-b)t}{y}, & \text{if } s = -t \neq 0,\ y \neq 0 \\ \beta & \text{if } s^2 \neq t^2, \end{cases}$$

where $A : R \to R$ is an additive function and a, b, c, α, β are arbitrary real constants.

Proof. To prove the theorem, we consider several cases depending on parameters s and t.

Case 1. Suppose $s = 0 = t$. Then (FE) yields

$$\frac{f(x) - g(y)}{x - y} = h(0)$$

which is

$$f(x) - ax = g(y) - ay,$$

where $a := h(0)$. From the above, we obtain

(3) $$f(x) = ax + b \quad \text{and} \quad g(y) = ay + b,$$

where b is an arbitrary constant. Letting (2) into (FE), we see that h is an arbitrary function with $a = h(0)$. Thus we obtain the solution as asserted in theorem for the case $s = 0 = t$.

Case 2. Suppose $s = 0$ and $t \neq 0$. Then from (FE), we get

(4) $$\frac{f(x) - g(y)}{x - y} = h(ty).$$

Putting $y = 0$ in (4), we see that

(5) $$f(x) = ax + b, \quad x \neq 0$$

where $a = h(0)$ and $b = g(0)$. Letting (5) into (4), we obtain

(6) $$ax + b - g(y) = (x - y)h(ty)$$

for all $x \neq y$ and $x \neq 0$. Equating the coefficients of x and the constant terms in (6), we get

(7) $$h(ty) = a \quad \text{and} \quad g(y) = h(ty)y + b = ay + b$$

for all $y \in R$. Letting $x = 0$ in (4) and using (7), we see that $f(0) = b$. Thus (5) holds for all x in $\mathbf{R}$. From (5) and (7), we get the solution of the (FE) for this case as asserted in Theorem 1.

Case 3. Suppose $s \neq 0 \neq t$. Letting $x = 0$ in (FE), we get

(8) $$g(y) = yh(ty) + b$$

for all $y \neq 0$ (where $b := f(0)$). Similarly, letting $y = 0$ in (FE), we get

(9) $$f(x) = xh(sx) + c$$

for all $x \neq 0$ (where $c := g(0)$). Inserting (8) and (9) into (FE) and simplifying, we obtain

(10) $$xh(sx) - yh(ty) + c - b = (x - y)h(sx + ty)$$

for all real nonzero x and y with $x \neq y$.

Replacing x by $\frac{x}{s}$ and y by $\frac{y}{t}$ in (10), we get

$$\frac{x}{s}h(x) - \frac{y}{t}h(y) + c - b = \left(\frac{x}{s} - \frac{y}{t}\right)h(x+y) \tag{11}$$

for all real nonzero x and y with $tx \neq sy$.

Subcase 3.1. Suppose $s = t$. Hence (11) yields

$$xh(x) - yh(y) = (b-c)t + (x-y)h(x+y). \tag{12}$$

Interchanging x with y in (12), we get $b = c$ and (12) reduces to

$$xh(x) - yh(y) = (x-y)h(x+y) \tag{13}$$

for all real nonzero x and y with $x \neq y$. Replacing y with $-y$ in (13), we obtain

$$xh(x) + yh(-y) = (x+y)h(x-y) \tag{14}$$

for all real nonzero x and y with $x + y \neq 0$. Letting $y = -x$ in (13), we see that

$$xh(x) + xh(-x) = 2xh(0). \tag{15}$$

Subtracting (14) from (13) and using (15), we get

$$2yh(0) = (x+y)h(x-y) - (x-y)h(x+y) \tag{16}$$

for all real nonzero x, y with $x + y$ and $x - y \neq 0$. Writing

$$u = x + y \quad \text{and} \quad v = x - y \tag{17}$$

in (16), we see that

$$(u-v)h(0) = uh(v) - vh(u)$$

which is

$$v[h(u) - h(0)] = u[h(v) - h(0)], \tag{18}$$

for all real nonzero $u, v, u - v$ and $u + v$. Thus

$$h(u) = \alpha u + a \tag{19}$$

for all real nonzero u in R (where $a := h(0)$). Notice that (19) also holds for $u = 0$. Using (19) in (FE), we get

$$f(x) - g(y) = (x-y)(\alpha tx + \alpha ty + a)$$

for all $x \neq y$. Thus, we obtain the asserted solution

$$f(x) = g(x) = \alpha tx^2 + ax + b \quad \text{and} \quad h(y) = \alpha y + a, \tag{20}$$

where α, a and b are arbitrary constants.

Subcase 3.2. Suppose $s = -t$. Then (11) yields

$$xh(x) + yh(y) + (b-c)t = (x+y)h(x+y) \tag{21}$$

for all real nonzero x and y with $x \neq y$. Define

$$A(x) = \begin{cases} xh(x) + (b-c)t & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases} \tag{22}$$

Then by (22), (21) reduces to

$$A(x) + A(y) = A(x+y) \tag{23}$$

for all real nonzero x, y and $x+y$. Next we show that A in (23) is additive on the set of reals. In order for A to be additive it must satisfy

$$\begin{cases} A(x) + A(-x) = A(0) = 0 \\ \text{or} \\ xh(x) - xh(-x) + 2(b-c)t = 0. \end{cases} \tag{24}$$

Interchanging y with $-y$ in (21), we get

$$xh(x) - yh(-y) + (b-c)t = (x-y)h(x-y). \tag{25}$$

Subtracting (25) from (21), we get

$$yh(y) + yh(-y) = (x+y)h(x+y) - (x-y)h(x-y).$$

Thus, using (22), we get

$$A(y) - A(-y) = A(x+y) - A(x-y) \tag{26}$$

for all real nonzero x, y, $x+y$ and $x-y$. Replacing x by $-x$ in (26), we obtain

$$A(y) - A(-y) = A(-x+y) - A(-x-y). \tag{27}$$

From (26) and (27), we get

$$A(x+y) + A(-(x+y)) = A(x-y) + A(-(x-y)). \tag{28}$$

Letting $u = x+y$ and $v = x-y$ in (28), we see that

$$A(u) + A(-u) = A(v) + A(-v)$$

for all real nonzero $u, v, u-v$ and $u+v$. Thus

$$A(u) + A(-u) = \gamma \tag{29}$$

for all real nonzero u (where γ is a constant). Using (22), we see from (29) that

$$xh(x) - xh(-x) + 2(b-c)t = \gamma, \tag{30}$$

for all real nonzero x. From (FE) with $s = -t$, we get

$$f(x) - g(y) = (x-y)h(-(x-y)t). \tag{31}$$

Interchanging x with y, we get

$$f(y) - g(x) = -(x-y)h((x-y)t). \tag{32}$$

Adding (31) to (32) and using (30), we get

(33) $f(x) - g(x) + f(y) - g(y) =$

$$= -(x-y)h((x-y)t) + (x-y)h(-(x-y)t) = -\frac{\gamma}{t} + 2(b-c).$$

Using (8) and (22), we obtain

(34) $$A(tx) = t[g(x) - c] \quad (x \neq 0).$$

Similarly, using (9) and (22), we get

(35) $$A(-tx) = -t[f(x) - b] \quad (x \neq 0).$$

So from (34) and (35), we see that

$$f(x) - g(x) = -\frac{A(-tx) + A(tx)}{t} + b - c = -\frac{\gamma}{t} + b - c.$$

Hence from above, we get

(36) $$f(x) - g(x) + f(y) - g(y) = -2\frac{\gamma}{t} + 2(b-c).$$

Comparing (33) with (36), we get $\gamma = 0$. Thus (29) yields

$$A(x) + A(-x) = 0,$$

for all real nonzero x. Evidently the above also holds for $x = 0$. Hence A is an additive function on the set of reals. From (22), (8) and (9), we obtain

(37) $$f(x) = \frac{A(tx)}{t} + b, \quad g(y) = \frac{A(ty)}{t} + c \quad \text{and}$$

$$h(y) = \frac{A(y)}{y} + \frac{(c-b)t}{y},$$

where b and c are arbitrary constants.

Subcase 3.3. Suppose $s^2 \neq t^2$, that is $s \neq \pm t$. Interchanging x with y in (11), we get

(38) $$\frac{y}{s}h(y) - \frac{x}{t}h(x) + c - b = \left(\frac{y}{s} - \frac{x}{t}\right)h(x+y)$$

for all nonzero x and y with $ty \neq sx$. Subtracting (38) from (11) and using $s + t \neq 0$, we get

(39) $$xh(x) - yh(y) = (x-y)h(x+y),$$

which is same as (13). Thus

(40) $$h(x) = \alpha x + b,$$

where α and b are arbitrary constants. Letting (40) into (38) and simplifying the resulting expression, we get

$$\alpha xy\left(\frac{1}{s}-\frac{1}{t}\right)=b-c$$

for all nonzero x and y with $tx \neq sy$ and $sx \neq ty$. Since $s \neq t$, we see that $\alpha = 0$ and $b = c$. Hence (40) becomes

$$h(x) = b. \tag{41}$$

From (41), (8) and (9), we obtained the asserted form of f, g and h. This completes the proof of the theorem.

Remark 1. In case of the functional equation (2) (that is when $g = f$), Subcase 3.2 simplifies to a great extent. If $g = f$, then the left side of the (FE) for $s = -t$ is symmetric in x and y. Thus using this symmetry one can conclude that h is an even function. The evenness of h implies that A in (23) is additive.

Remark 2. In Subcase 3.1, $h(y)$ is undefined at $y = 0$.

Remark 3. It is well known that the functional equation $A(x + y) = A(x) + A(y)$ has nonmeasurable solutions in addition to the continuous solution of the form $A(x) = ax$, where a is an arbitrary real constant. Since, additive function appears in the solution of (FE) for Subcase $s = -t$, it follows that (FE) has non-measurable solutions. However, all measurable solutions of (FE) are continuous and polynomials of low degree.

The following theorem is obvious from the Theorem 1.

THEOREM 2. *Functions $\phi, f : R \to R$ satisfy functional equation (2) for all $x, y \in R$ with $x \neq y$ if and only if*

$$f(x)=\begin{cases} ax+b & \text{if } s=0=t\\ ax+c & \text{if } s=0,\ t\neq 0\\ \alpha t x^2+ax+b & \text{if } s=t\neq 0\\ \dfrac{A(tx)}{t}+b, & \text{if } s=-t\neq 0\\ \beta x+b & \text{if } s^2\neq t^2\end{cases}$$

$$\phi(y)=\begin{cases} \text{arbitrary with } \phi(0)=a & \text{if } s=0=t\\ a & \text{if } s=0,\ t\neq 0\\ \alpha y+y & \text{if } s=t\neq 0\\ \dfrac{A(y)}{y}, & \text{if } s=-t\neq 0,\ y\neq 0\\ \beta & \text{if } s^2\neq t^2,\end{cases}$$

where $A : R \to R$ *is an additive function and* a, b, c, α, β *are arbitrary real constants.*

The following corollary addresses the problem proposed by Walter Rudin in [11].

COROLLARY 3. *The function* $f : R \to R$ *satisfies the equation*

$$f'(sx + ty) = \frac{f(y) - f(x)}{y - x}$$

for all $x, y \in R$ *with* $x \neq y$ *if and only if*

$$f(x) = \begin{cases} ax^2 + bx + c & \text{if } s = \frac{1}{2} = t \\ bx + c & \text{otherwise,} \end{cases}$$

where a, b *and* c *are arbitrary real constants.*

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ASYMPTOTIC PROPERTIES OF SOLUTIONS OF HIGHER ORDER LINEAR DIFFERENCE EQUATIONS

1. Introduction

In this paper we will investigate the asymptotic behaviour of solutions of certain forms of m-th order linear difference equations. Motivated by the work of Trench [4] on differential equations we will give some conditions under which the equation

$$\text{(E)} \qquad \Delta^m x_n + p_{1,n}\Delta^{m-1}x_n + \ldots + p_{m-1,n}\Delta x_n + p_{m,n}x_n = f_n, \qquad n \in N, m \geq 2,$$

where $p_1, \ldots, p_m, f : N \to R$ has a solution x_0 behaving for a sufficiently large n like a given polynomial q of degree $< m$.

Let us use the following notations: $x_n = x(n), R := (-\infty, \infty), R_+ := (0, \infty), N := \{n_0, n_0 + 1, \ldots\}$, where n_0 is a given nonnegative integer. For a function $x : N \to R$, we define the difference operator Δ^i as follows

$$\Delta^0 x_n = x_n, \Delta^k x_n = \Delta(\Delta^{k-1}x_n) = \Delta^{k-1}x_{n+1} - \Delta^{k-1}x_n, \quad k \geq 1.$$

Further, by $n^{(k)}$ we will denote the product

$$n^{(k)} = \prod_{j=0}^{k-1}(n-j) \quad \text{for } n \geq k, \; n^{(0)} = 1,$$

where k is a positive integer.

2. Main result

Throughout this paper q is a given polynomial of degree $< m$. For convenience, we note

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(1) $$Mx_n = \sum_{k=1}^{m} p_{k,n} \Delta^{m-k} x_n.$$

Hence, the equation (E) can be rewritten in the form

(2) $$\Delta^m x_n + M x_n = f_n.$$

Now, we introduce the new unknown h defined by the formula

(3) $$h = x - q.$$

Because $\Delta^m q_n = 0$, it is obvious that x is a solution of (2) if and only if h is a solution of

(4) $$\Delta^m h_n = -M h_n - g_n,$$

where

(5) $$g_n = -f_n + M q_n = -f_n + \sum_{k=1}^{m} p_{k,n} \Delta^{m-k} q_n.$$

LEMMA. *Let $\Phi : N \to R_+$ be a nonincreasing sequence and $u : N \to R$. Let us assume that the series*

(6) $$\sum_{j=n_0}^{\infty} j^{m-1} u_j$$

is convergent and that

(7) $$\sum_{j=n}^{\infty} j^{m-1} u_j = O(\Phi_n).$$

Further, let us define

(8) $$\varrho_n = \sup_{\tau \ge n} \Big| \Phi_\tau^{-1} \sum_{j=\tau}^{\infty} j^{m-1} u_j \Big|.$$

Then for

(9) $$w_n = (-1)^{m-1} \sum_{j=n}^{\infty} \frac{(j+m-1-n)^{(m-1)}}{(m-1)!} u_j$$

the inequalities

(10) $$|\Delta^r w_n| \le \frac{2\varrho_n \Phi_n n^{-r}}{(m-1-r)!}, \quad 0 \le r \le m-1,$$

are satisfied. Moreover, if

(11) $$\lim_{n\to\infty} \varrho_n = 0,$$

then

(12) $$\Delta^r w_n = o(\Phi_n n^{-r}), \quad 0 \le r \le m-1.$$

The proof of this Lemma is in section 3.

Now we can formulate the main result of the paper.

THEOREM. *Let us assume that series*

$$\sum_{j=n_0}^{\infty} j^{m-1} g_j \tag{13}$$

is convergent and that

$$\sum_{j=n}^{\infty} j^{m-1} g_j = O(\Phi_n), \tag{14}$$

where g is defined by (5) and Φ_n is a nonincreasing positive sequence. Moreover, let us assume that

$$\sum_{j=n_0}^{\infty} j^{k-1} |p_{k,j}| < \infty, \quad 1 \le k \le m. \tag{15}$$

Then equation (E) *has a solution x_0 such that*

$$\Delta^r x_0 = \Delta^r q_n + O(\Phi_n n^{-r}), \quad 0 \le r \le m-1. \tag{16}$$

Proof. Let us denote by $m(\Phi)$ the Banach space of sequences $h : N \to R$ satisfying the condition

$$\Delta^r h_n = O(\Phi_n n^{-r}), \quad 0 \le r \le m-1 \tag{17}$$

with the norm

$$\|h\| = \sup_{n \ge n_0} \left\{ \Phi_n^{-1} \sum_{r=0}^{m-1} n^r |\Delta^r h_n| \right\}. \tag{18}$$

We will show that the equation (4) has a solution with expected property and that it is a fix point of a contraction mapping of space $m(\Phi)$ into itself.

Let us note

$$\overline{n} = \sup \left\{ t \in N : \sum_{k=1}^{m} \sum_{j=t}^{\infty} j^{k-1} |p_{k,j}| \ge \sum_{r=0}^{m-1} \frac{(m-r-1)!}{4} \right\}, \tag{19}$$

Now we define the transformation L by

$$(Lh) = \begin{cases} 0, & n < \overline{n} \\ (-1)^{m-1} \sum_{j=n}^{\infty} \frac{(j+m-1-n)^{(m-1)}}{(m-1)!} M h_j, & n \ge \overline{n}. \end{cases} \tag{20}$$

With g as in (4), let

$$s_n = (-1)^{m-1} \sum_{j=n}^{\infty} \frac{(j+m-1-n)^{(m-1)}}{(m-1)!} g_n. \tag{21}$$

We will show that the mapping T define by

$$Th = s + Lh \tag{22}$$

maps $m(\Phi)$ into itself and is a contraction mapping.

At first we must prove that the sequences (20) and (21) are convergent. In accordance with (1), one can write

$$\text{(23)}\qquad \sum_{j=n}^{\infty}\frac{(j+m-1-n)^{(m-1)}}{(m-1)!}Mh_j = \\ =\sum_{j=n}^{\infty}\frac{(j+m-1-n)^{(m-1)}}{(m-1)!}\Big[\sum_{k=1}^{m}p_{k,j}\Delta^{m-k}h_j\Big].$$

In the following, let $j \geq n \geq n_0 \geq m-1$. From (18) we get

$$\Phi_j^{-1}\sum_{r=0}^{m-1} j^r|\Delta^r h_j| \leq \|h\|.$$

Hence, for each $r = 0,1,\ldots,m-1$, we have $\Phi_j^{-1}j^r|\Delta^r h_j| \leq \|h\|$. Taking $r = m-k$, we obtain

$$j^{m-1}|\Delta^{m-k}h_j| \leq \|h\|\Phi_j j^{k-1}, \qquad 1 \leq k \leq m.$$

From the above inequality and from monotonicity of Φ we get

$$\text{(24)}\qquad \sum_{j=n}^{\infty} j^{m-1}|\Delta^{m-k}h_j||p_{k,j}| \leq \Phi_n\|h\|\sum_{j=n}^{\infty} j^{k-1}|p_{k,j}|, \qquad 1 \leq k \leq m,$$

and, by virtue of (15), we have the convergence of the series on the left-hand side of (24).

Let us consider the series on the right-hand side of (23). From (24) it follows that the series $\sum_{j=n}^{\infty} j^{m-1}p_{k,j}\Delta^{m-k}h_j$ is absolutely convergent. Convergent is the series $\sum_{k=1}^{m} j^{m-1}[\sum_{j=n}^{\infty} p_{k,j}\Delta^{m-k}h_j]$, too. So, by virtue of Abel's criterion, the series

$$\sum_{j=n}^{\infty}\frac{(j+m-1-n)^{(m-1)}}{(m-1)!}\Big[\sum_{k=1}^{\infty}p_{k,j}\Delta^{m-k}h_j\Big].$$

is convergent. Furthermore, the equality

$$\sum_{j=n}^{\infty}\frac{(j+m-1-n)^{(m-1)}}{(m-1)!}\Big[\sum_{k=1}^{m}p_{k,j}\Delta^{m-k}h_j\Big] = \\ =\sum_{j=n}^{\infty}\sum_{k=1}^{m}\frac{(j+m-1-n)^{(m-1)}}{(m-1)!}p_{k,j}\Delta^{m-k}h_j$$

holds. The convergence of the series (21) follows from Lemma for $u = g$.

Noting

$$I(n;h) = \sum_{k=1}^{m}\sum_{j=n}^{\infty} j^{m-1}p_{k,j}\Delta^{m-k}h_j,$$

by (24), we get the estimate

(25) $$|I(n;h)| \leq \Phi_n \|h\| \sigma_n,$$

where $\sigma_n = \sum_{k=1}^{m} \sum_{j=n}^{\infty} j^{k-1} |p_{k,j}|$.

Now, we can apply Lemma with $u = Mh$ and $w = Lh$. Then (8) becomes $\varrho_n = \sup_{\tau \geq n} \Phi_\tau^{-1} |I(\tau; h)|$ which, with (25), implies that

(26) $$\varrho_n \leq \|h\| \sup_{\tau \geq n} \sigma_\tau = o(1).$$

Further, we have (applying (10))

$$\Phi_n^{-1} n^r |\Delta^r (Lh)_n| \leq \frac{2\varrho_n}{(m-1-r)!}, \quad r = 0, 1, \ldots, m-1.$$

From the above and from (26) it follows that $Lh \in m(\Phi)$ and

(27) $$\|Lh\| \leq K \|h\| \sup_{\tau \geq n_0} \sigma_\tau,$$

where $K = \sum_{r=0}^{m-1} \frac{2}{(m-r-1)!}$ is an universal constant.

By virtue of Lemma (for $u = g$), we have the inequality

$$|\Delta^r s_n| \leq \frac{2\varrho_n \Phi_n n^{-r}}{(m-r-1)!}$$

implying $s \in m(\Phi)$, by the assumption (14). Hence, the transformation T defined by (22) is a mapping of $m(\Phi)$ into itself.

Let $h_1, h_2 \in m(\Phi)$. Then, using (19) and (27), we have

$$\|Th_1 - Th_2\| = \|L(h_1 - h_2)\| \leq K \|h_1 - h_2\| \sup_{\tau \geq n_0} \sigma_\tau = \frac{1}{2} \|h_1 - h_2\|.$$

Hence, T is a contraction mapping of $m(\Phi)$ into itself. So it has a fixed point h_0 such that $Th_0 = h_0$. From (20) and (21) it follows that h_0 satisfies the equation

$$h_{0,n} = (-1)^{m-1} \sum_{j=n}^{\infty} \frac{(j+m-1-n)^{(m-1)}}{(m-1)!} (Mh_{0,j} + g_j).$$

Hence $\Delta^{m-1} h_{0,n} = \sum_{j=n}^{\infty} (Mh_{0,j} + g_j)$, and it means that $\Delta^m h_{0,n} = -Mh_{0,n} - g_n$. So, h_0 is a solution of the equation (4), too. Since $h_0 \in m(\Phi)$, the condition (17) is satisfied. From this fact and from (3) we get the condition (18) and this completes the proof of Theorem.

COROLLARY 1. *If in assumption* (14) *of Theorem we interchange „O" with „o", then Theorem still holds for* (16) *with „O" interchanged with „o".*

Proof. Since from (26) follows (11), from Lemma with $u = Mh$ and $w = Lh_0$ we get

(28) $$\Delta^r Lh_{0,n} = o(\Phi_n n^{-r}), \quad 0 \leq r \leq m-1.$$

Moreover, if in (14) we change „O" with „o", then from Lemma it follows

$$\Delta^r s_n = o(\Phi_n n^{-r}), \quad 0 \le r \le m-1. \tag{29}$$

But $h_0 = s + Lh_0$, so, by means of (28) and (29), it follows that (17) can be replaced by $\Delta^r h_n = o(\Phi_n n^{-r})$.

EXAMPLE. Let us assume that the series

$$\sum_{j=n_0}^{\infty} j^{k-1}|p_{k,j}|, \quad 1 \le k \le m, \tag{30}$$

are convergent. Let v be any integer in $\{0, 1, \ldots, m-1\}$ and let $q_n = \frac{n^{(v-r)}}{v!}$, $f_n \equiv 0$. Then the function g defined by the formula (5) has the form

$$g_n = \sum_{k=m-v}^{\infty} p_{k,n} \frac{n^{(v-m+k)}}{(v-m+k)!}.$$

From (28) it follows that

$$\sum_{j=n}^{\infty} j^{m-1} g_j = O(\Phi_n),$$

where $\Phi_n = n^{-v}$. By virtue of Theorem, the homogeneous equation

$$\Delta^m x_n + p_{1,n}\Delta^{m-1}x_n + \ldots + p_{m-1,n}\Delta x_n + p_{m,n}x_n = 0$$

has a fundamental system of solutions $x_0, x_1, \ldots, x_{m-1}$ such that

$$\Delta^r x_{v,n} = \begin{cases} \frac{n^{(v-r)}}{(v-r)!}, & 0 \le r \le v, \\ O(n^{v-r}), & v+1 \le r \le m-1. \end{cases}$$

3. Appendix. Proof of Lemma

Let us denote

$$Q_n = \sum_{j=n}^{\infty} j^{m-1} u_j. \tag{31}$$

From definition of the difference we get $\Delta Q_n = -n^{m-1}u_n$. Hence

$$u_n = -\frac{\Delta Q_n}{n^{m-1}}. \tag{32}$$

By virtue of (8) and (31), we have $\varrho_n \ge \Phi_n^{-1}|Q_n|$. So, we obtain

$$\Phi_n \varrho \ge |Q_n|. \tag{33}$$

By (6), the series $\sum_{j=n}^{\infty} j^{m-1}u_j$, is convergent. The sequence

$$\left\{\frac{(j+m-1-n)^{(m-1)}}{j^{m-1}}\right\}_{j=n}^{\infty}$$

is increasing and bounded; hence, by virtue of Abel's criterion, the series

$$\sum_{j=n}^{\infty} \frac{(j+m-1-n)^{(m-1)}}{(m-1)!} u_j = \frac{1}{(m-1)!} \sum_{j=n}^{\infty} j^{m-1} u_j \frac{(j+m-1-n)^{(m-1)}}{j^{m-1}}$$

is convergent, too.

Using definition (9) and the definition of difference Δ, we get

$$\Delta^r w_n = (-1)^{m-1-r} \sum_{j=1}^{\infty} \frac{(j+m-1-n-r)^{(m-1-r)}}{(m-1-r)!} u_j$$

for $j \geq n$ and $0 \leq r \leq m-1$.

The series $\sum_{j=n}^{\infty} (j+m-1-n-r)^{(m-1-r)} u_j$ can be written in the form

$$\sum_{j=n}^{\infty} (j+m-1-n-r)^{(m-1-r)} u_j = \tag{34}$$

$$= \sum_{j=n}^{\infty} \Big[\prod_{t=-m+1+n+r}^{n-1} \Big(1 - \frac{t}{j}\Big)\Big] j^{m-r-1} u_j.$$

The series on the left-hand side of (34) is convergent; hence, that on the right-hand side is convergent, too.

Applying (32), we can write (34) in the form

$$\sum_{j=n}^{\infty} (j+m-1-n-r)^{(m-1-r)} u_j = \sum_{j=n}^{\infty} \Big[\prod_{t=-m+1+n+r}^{n-1} \Big(1 - \frac{t}{j}\Big)\Big] j^{-r} (-\Delta Q_j).$$

Further, applying the formula for summing by parts, we get

$$\sum_{j=n}^{\infty} (j+m-1-n-r)^{(m-1-r)} u_j = \tag{35}$$

$$= \lim_{s\to\infty} \sum_{j=n}^{s} (j+m-1-n-r)^{(m-1-r)} u_j$$

$$= -\lim_{s\to\infty} (s+1)^{-r} Q_{s+1} \prod_{t=-m+1+n+r}^{n-1} \Big(1 - \frac{t}{s+1}\Big) +$$

$$+ n^{-r} Q_n \prod_{t=-m+1+n+r}^{n-1} \Big(1 - \frac{t}{n}\Big) +$$

$$+ \sum_{j=n}^{\infty} Q_{j+1} \Delta \Big[j^{-r} \prod_{t=-m+1+n+r}^{n-1} \Big(1 - \frac{t}{j}\Big)\Big].$$

Since, by virtue of (6), we have $Q_{s+1} \to 0$, as $s \to \infty$, and the sequences $\{(s+1)^{-r}\}$ and $\{\prod_{t=-m+1+n+r}^{n-1}(1-\frac{t}{s+1})\}_{s=n}^{\infty}$ are bounded, so we have

$$\lim_{s\to\infty}(s+1)^{-r}Q_{s+1}\Big[\prod_{t=-m+1+n+r}^{n-1}\Big(1-\frac{t}{s+1}\Big)\Big]=0, \tag{36}$$

and (35) can be rewritten in the form

$$\sum_{j=n}^{\infty}(j+m-1-n-r)^{(m-r-1)}u_j =$$
$$= n^{-r}\prod_{t=-m+1+n+r}^{n-1}\Big(1-\frac{t}{n}\Big)Q_n+\sum_{j=n}^{\infty}\Delta\Big[j^{-r}\prod_{t=-m+1+n+r}^{n-1}\Big(1-\frac{t}{j}\Big)\Big]Q_{j+1},$$

And again, because the series on the left-hand side is convergent, so that on the right-hand side of (35) is convergent, too.

Applying the formula for the difference of a product, we can rewrite (35) in the form

$$\sum_{j=n}^{\infty}(j+m-1-n-r)^{(m-1-r)}u_j = n^{-r}\prod_{t=-m+1+n+r}^{n-1}\Big(1-\frac{t}{n}\Big)Q_n+ \tag{37}$$
$$+\sum_{j=n}^{\infty}Q_{j+1}\prod_{t=-m+1+n+r}^{n-1}\Big(1-\frac{t}{j+1}\Big)\Delta j^{-r}+$$
$$+\sum_{j=n}^{\infty}j^{-r}\Delta\Big[\prod_{t=-m+1+n+r}^{n-1}\Big(1-\frac{t}{j}\Big)\Big]Q_{j+1}.$$

One can observe that

$$\sum_{j=n}^{\infty}\Big[\Delta\prod_{t=-m+1+n+r}^{n-1}\Big(1-\frac{t}{j}\Big)\Big]=\lim_{s\to\infty}\sum_{j=n}^{s}\Delta\Big[\prod_{t=-m+1+n+r}^{n-1}\Big(1-\frac{t}{j}\Big)\Big]=$$
$$=\lim_{s\to\infty}\Big[\prod_{t=-m+1+n+r}^{n-1}(1-\frac{t}{s+1})-\prod_{t=-m+1+n+r}^{n-1}\Big(1-\frac{t}{n}\Big)\Big]=$$
$$=1-\prod_{t=-m+1+n+r}^{n-1}(1-\frac{t}{n}).$$

Hence, the series $\sum_{j=n}^{\infty}\Delta[\prod_{t=-m+1+n+r}^{n-1}(1-\frac{t}{j})]$ is convergent. Its terms being positive, it is absolutely convergent. The sequence $\{j^{-1}Q_{j+1}\}$ is

bounded, so the series

$$\sum_{j=n}^{\infty} j^{-r} Q_{j+1} \Delta \Big[\prod_{t=-m+1+n+r}^{n-1} \Big(1 - \frac{t}{j}\Big) \Big]$$

is absolutely convergent.

From equality (37) it follows that the series

$$\sum_{j=n}^{\infty} Q_{j+1} \prod_{t=-m+1+n+r}^{n-1} \Big(1 - \frac{t}{j+1}\Big) \Delta j^{-r}$$

is convergent, too. Further, applying (37), we have

$$\Big| \sum_{j=n}^{\infty} (j+m-1-n-r)^{(m-1-r)} u_j \Big| \le n^{-r} \prod_{t=-m+1+n+r}^{n-1} \Big(1 - \frac{t}{n}\Big) |Q_n| +$$
$$+ \sum_{j=n}^{\infty} |Q_{j+1}| \Big| \prod_{t=-m+1+n+r}^{n-1} \Big(1 - \frac{t}{j+1}\Big) \Big| |\Delta j^{-r}| +$$
$$+ \sum_{j=n}^{\infty} j^{-r} \Big| \Delta \Big[\prod_{t=-m+1+n+r}^{n-1} \Big(1 - \frac{t}{j}\Big) \Big] \Big| |Q_{j+1}|.$$

Since the sequence $\{|Q_n|\}_{n=n_0}^{\infty}$ is bounded from above (see (33)) by a nonincreasing (see (8)) sequence $\{\Phi_n \varrho_n\}$, we have $|Q_j| \le \varrho_n \Phi_n$ and $|Q_{j+1}| \le \varrho_{j+1} \Phi_{j+1} \le \varrho_n \Phi_n$ for $j \ge n$. The sequence $\{\prod_{t=-m+1+n+r}^{n-1} (1 - \frac{t}{j}\}_{j=n}^{\infty}$ is nondecreasing positive and bounded from above by 1. Furthermore $\{n^{-r}\}_{n=n_0}^{\infty}$ is a positive nonincreasing sequence for $0 \le r \le m-1$, so we get

$$\Big| \sum_{j=n}^{\infty} (j+m-1-n-r)^{(m-1-r)} u_j \Big| \le$$
$$\le n^{-r} \Phi_n \varrho_n \Big[\prod_{t=-m+1+n+r}^{n-1} \Big(1 - \frac{t}{n}\Big) \Big] +$$
$$+ \sum_{j=n}^{\infty} |\Delta j^{-r}| \Phi_n \varrho_n \Big[\prod_{t=-m+1+n+r}^{n-1} \Big(1 - \frac{t}{j+1}\Big) \Big] +$$
$$+ \sum_{j=n}^{\infty} j^{-r} \Phi_n \varrho_n \Delta \Big[\prod_{t=-m+1+n+r}^{n-1} \Big(1 - \frac{t}{j}\Big) \Big] \le$$
$$\le n^{-r} \Phi_n \varrho_n \Big[\prod_{t=-m+1+n+r}^{n-1} \Big(1 - \frac{t}{n}\Big) \Big] + \Phi_n \varrho_n \Big[- \sum_{j=n}^{\infty} \Delta j^{-r} \Big] +$$

$$+ n^{-r}\Phi_n\varrho_n \sum_{j=n}^{\infty} \Delta\Big[\prod_{t=-m+1+n+r}^{n-1}\Big(1-\frac{t}{j}\Big)\Big] =$$

$$= n^{-r}\Phi_n\varrho_n\Big\{\prod_{t=-m+1+n+r}^{n-1}(1-\frac{t}{n})+1+1+$$

$$-\prod_{t=-m+1+n+r}^{n-1}\Big(1-\frac{t}{n}\Big)\Big\} = 2n^{-r}\Phi_n\varrho_n.$$

Finally, we obtain

$$|\Delta^r w_n| \le \frac{1}{(m-1-r)!}\Big|\sum_{j=n}^{\infty}(j+m-1-n-r)^{(m-1-r)}u_j\Big| \le \frac{2n^{-r}\Phi_n\varrho_n}{(m-1-r)!}$$

for $0 \le r \le m-1$. Now, in order to prove (12), we state that

$$\frac{|\Delta^r w_n|}{\Phi_n n^{-r}} \le \frac{2\varrho_n}{(m-r-1)!},$$

because the sequence Φ_n is positive and $n > 0$. By the assumption (11) of our Lemma, we get $\lim_{n\to\infty}\varrho_n = 0$.

Hence

$$\frac{|\Delta^r w_n|}{\Phi_n n^{-r}} \to 0, \quad \text{as } n \to \infty,$$

so $\Delta^r w_n = o(\Phi_n n^{-r})$ for $0 \le r \le m-1$. This completes the proof of Lemma.

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ON LUSIN'S THEOREM IN THE ASPECT OF SMALL SYSTEMS

The main purpose of this article is to consider Lusin's theorem in the abstract sense, formulated in terms of some systems of "small" sets without using any measure. The aspect of category and measure of Lusin's theorem is also included. Theorem 1 presented here has a more general version than the analogous one in [1].

Let X denote a nonempty abstract set and $\mathcal{S}$—a σ-field of subsets of X.

DEFINITION 1. (cf. [1]). We shall say that a sequence $\{\mathcal{N}_n\}_{n\in N}$ of subfamilies of $\mathcal{S}$ is a small system on $\mathcal{S}$ if:

(1) $\emptyset \in \mathcal{N}_n$ for each $n \in N$,

(2) for any $n \in N$, there exists a sequence $\{k_i\}_{i\in N}$ of positive integers such that if $A_i \in \mathcal{N}_{k_i}$ for $i \in N$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{N}_n$,

(3) for any $n \in N$, $A \in \mathcal{N}_n$ and $B \in \mathcal{S}$, such that $B \subset A$, we have $B \in \mathcal{N}_n$,

(4) for any $n \in N$, $A \in \mathcal{N}_n$ and $B \in \bigcap_{i=1}^{\infty} \mathcal{N}_n$, we have $A \cup B \in \mathcal{N}_n$,

(5) $\mathcal{N}_{n+1} \subset \mathcal{N}_n$ for each $n \in N$.

It is easy to check that the family $\mathcal{N} = \bigcap_{n=1}^{\infty} \mathcal{N}_n$ forms a σ-ideal of $\mathcal{S}$-measurable sets, i.e. $\mathcal{N}$ is closed under countable unions and any $\mathcal{S}$-measurable subset of a set from $\mathcal{N}$ is a member of $\mathcal{N}$. Further, $\mathcal{N}$ will denote the σ-ideal $\bigcap_{n=1}^{\infty} \mathcal{N}_n$.

Let X and Y be topological spaces, let $\mathcal{S}$ denote a σ-field of subsets of X containing the family of Borel sets $\mathcal{B}(X)$ and let $\{\mathcal{N}_n\}_{n\in N}$ be a small system on $\mathcal{S}$.

DEFINITION 2. We shall say that a function $f : X \to Y$ fulfils Lusin's condition with respect to the small system $\{\mathcal{N}_n\}_{n\in N}$ (in abbr. $(\mathcal{N}_n)$-Lusin condition) if, for each positive integer n_0, there exists a closed set D_{n_0} such that $f_{|D_{n_0}}$ is continuous and $X - D_{n_0} \in \mathcal{N}_{n_0}$.

DEFINITION 3. We shall say that a function $f : X \to Y$ fulfils the weak Lusin condition with repsect to the small system $\{\mathcal{N}_n\}_{n\in N}$ (in abbr. $\{\mathcal{N}_n\}$—weak Lusin condition) if, for each positive integer n_0, there exists a measurable set A_{n_0} such that $f_{|A_{n_0}}$ is continuous and $X - A_{n_0} \in \mathcal{N}_{n_0}$.

It is clear that any $\mathcal{S}$-measurable function does not have to satisfy even the $(\mathcal{N}_n)$—weak Lusin condition. For example, if $\mathcal{S} = 2^x$ and $\mathcal{N}_n = \{\emptyset\}$ for every $n \in N$, then every non-continuous function $f : X \to Y$ does not fulfil the $(\mathcal{N}_n)$—weak Lusin condition.

But let us pay attention to the following property.

PROPOSITION. *If each subset of a set belonging to $\mathcal{N}$ is $\mathcal{S}$-measurable, then a function $f : X \to Y$ fulfilling the $(\mathcal{N}_n)$—weak Lusin condition is $\mathcal{S}$-measurable.*

Proof. For every positive integer n there exists an $\mathcal{S}$-measurable set A_n such that $f_{|A_n}$ is continuous and $X - A_n \in \mathcal{N}_n$. Let V be any open subset of Y. By the following equality

$$f^{-1}(V) = \bigcup_{n=1}^{\infty} f^{-1}_{|A_n}(V) \cup \bigcap_{n=1}^{\infty} f^{-1}(V) \cap (X - A_n)$$

we conclude that $f^{-1}(V)$ is $\mathcal{S}$-measurable.

DEFINITION 4. (cf. [3]). We shall say that a small system $\{\mathcal{N}_n\}_{n\in N}$ is regular if, for any set $A \in \mathcal{S}$ and any positive integer n, there exists a closed set $F \in \mathcal{S}$ such that $F \subset A$ and $A - F \in \mathcal{N}_n$.

DEFINITION 5. (cf.[3]). We shall say that a small system $\{\mathcal{N}_n\}_{n\in N}$ is weakly regular if, for any set $A \in \mathcal{S}$ and any positive integer n, there exists a closed set $F \in \mathcal{S}$ such that $A \triangle F \in \mathcal{N}_n$.

It was observed (see Proposition 1.3(b) in [3]) that, instead of the closed set F, we can take an open set in Definition 5.

THEOREM 1. *Let Y be a topological antidiscrete space with a countable net of Borel sets. Then a small system $\{\mathcal{N}_n\}_{n\in N}$ on $\mathcal{S}$ is weakly regular if and only if any $\mathcal{S}$-measurable function $f : X \to Y$ fulfils the $(\mathcal{N}_n)$—weak Lusin condition.*

Proof. Necessity. Let us fix $n \in N$ and take a sequence (k_i) of positive integers such that $\bigcup_{i=1}^{\infty} A_i \in \mathcal{N}_n$ whenever $A_i \in \mathcal{N}_{k_i}$ for $i \in N$. Let $\{E_1, E_2, \ldots\}$ be a countable net of Borel sets in Y. Since f is $\mathcal{S}$-measurable and $\{\mathcal{N}_n\}_{n\in N}$ is weakly regular, for each $i \in N$, there exists an open set G_i with $f^{-1}(E_i) \triangle G_i \in \mathcal{N}_{k_i}$. Put

$$A = \bigcap_{i=1}^{\infty} [X - (f^{-1}(E_i) \triangle G_i)].$$

Then

$$X - A = \bigcup_{i=1}^{\infty}(f^{-1}(E_i)\triangle G_i)$$

belongs to $\mathcal{N}_n$. Moreover, for any $i_0 \in N$, $f^{-1}(E_{i_0}) \cap A = G_{i_0} \cap A$. Indeed, if $x \in f^{-1}(E_{i_0}) \cap A$, then

$$\begin{aligned} x &\in f^{-1}(E_{i_0}) - [f^{-1}(E_{i_0}) \triangle G_{i_0}] \\ &= f^{-1}(E_{i_0}) - [(f^{-1}(E_{i_0}) - G_{i_0}) \cup (G_{i_0} - f^{-1}(E_{i_0}))] \\ &= [f^{-1}(E_{i_0}) - [f^{-1}(E_{i_0}) - G_{i_0}]] \\ &\cap [f^{-1}(E_{i_0}) - (G_{i_0} - f^{-1}(E_{i_0})] = f^{-1}(E_{i_0}) \cap G_{i_0}; \end{aligned}$$

on the other hand, if $x \in G_{i_0} \cap A$, then

$$x \in G_{i_0} \cap [X - (f^{-1}(E_{i_0}) \triangle G_{i_0})] = G_{i_0} \cap f^{-1}(E_{i_0}),$$

so, $f^{-1}(E_{i_0}) \cap A = G_{i_0} \cap A$ is an open subset of A. If U is open in Y, then U is a union of some subfamily of $\{E_1, E_2, \ldots\}$, therefore $f^{-1}(U)$ is open in A. Consequently, $f_{|A}$ is continuous.

Sufficiency. Let $A \in \mathcal{S}$. If $A = X$, then the condition of weak regularity is fulfilled. Suppose that $X - A \neq \emptyset$. Let V be a proper nonempty open set in Y. Let $y_1 \neq y_2$ be arbitrary points of Y such that $y_1 \in V$ and $y_2 \notin V$. Let us consider the following function

$$f(x) = \begin{cases} y_1, & x \in A \\ y_2, & x \notin A. \end{cases}$$

It is obvious that f is $\mathcal{S}$-measurable. Let us fix $n \in N$. Since f fulfils the $(\mathcal{N}_n)$—weak Lusin condition, there exists a measurable set $B \subset X$ such that $f_{|B}$ is continuous and $X - B \in \mathcal{N}_n$. The set $f_{|B}^{-1}(V) = A \cap B$ is open in B, thus there exists an open set $W \subset X$ such that $W \cap B = A \cap B$.

Observe that $A\triangle W \subset X - B$.

Hence $A \triangle W \in \mathcal{N}_n$, which ends the proof of the weak regularity of the small system $\{\mathcal{N}_n\}_{n\in N}$.

If $\mathcal{N}_n$ denotes the family of subsets of the first category in X for each positive integer n, then it is clear that the small system $\{\mathcal{N}_n\}_{n\in N}$ is weakly regular on the σ-field of sets having the Baire property. Thus, by Proposition 1 and Theorem 2 we can formulate well-known category aspect of Lusin's theorem (see [2]).

COROLLARY 1. *A function $f : X \to R$ has the Baire property if and only if there exists a set A having the Baire property, such that $f_{|A}$ is continuous and $X - A$ is the set of the first category.*

THEOREM 2. *Let Y be a topological antidiscrete space with a countable net of Borel sets. Then a small system $\{\mathcal{N}_n\}_{n\in N}$ on $\mathcal{S}$ is regular if and only if any $\mathcal{S}$-measurable function $f : X \to Y$ fulfils the $(\mathcal{N}_n)$—Lusin condition.*

Proof. Necessity. Let us fix $n \in N$ and let n_1, n_2, be positive integers such that if $A_1 \in \mathcal{N}_{n_1}$ and $A_2 \in \mathcal{N}_{n_2}$, then $A_1 \cup A_2 \in \mathcal{N}_n$. It is obvious that a regular system is weakly regular, so, by the previous theorem, there exists an $\mathcal{S}$-measurable set A such that $X - A \in \mathcal{N}_{n_1}$ and $f_{|A}$ is continuous. On the other hand, by the regularity property of the small system, there exists a closed set $F \subset A$ such that $A - F \in \mathcal{N}_{n_2}$. Since $X - F \subset (X - A) \cup (A - F)$, we conclude that $X - F \in \mathcal{N}_n$ and, clearly, $f_{|F}$ is continuous.

Sufficiency. Let A be an arbitrary $\mathcal{S}$-measurable set such that $X - A \neq \emptyset$. Let V be any open proper set in Y and $y_1 \neq y_2$ be arbitrary points of Y such that $y_1 \in V$ and $y_2 \notin V$. Let us define $\mathcal{S}$-measurable function f

$$f(x) = \begin{cases} y_1, & x \notin A, \\ y_2, & x \in A. \end{cases}$$

Then, for an arbitrary positive integer n, there exists a closed set D such that $f_{|D}$ is continuous and $X - D \in \mathcal{N}_n$. Hence $f_{|D}^{-1}(Y - V) = A \cap D$ is a closed set in the subspace D thus it is closed in X. Simultaneously, $A - (A \cap D) = A \cap (X - D) \in \mathcal{N}_n$, which ends the proof of the regularity of the small system $\{\mathcal{N}_n\}_{n \in N}$.

Let γ be any finite inner regular measure over X and $\mathcal{S}$ denote the σ-field of all γ-measurable subsets of X containing $\mathcal{B}(X)$. Putting $\mathcal{N}_n = \{A \in \mathcal{S} : \gamma(A) \leq \frac{1}{n}\}$ we obtain a small system $\{\mathcal{N}_n\}_{n \in N}$ on $\mathcal{S}$ which is regular. Thus by Proposition and Theorem 2 we conclude the classical Lusin theorem.

Corollary 2. *A function $f : X \to R$ is γ-measurable if and only if, for every $\varepsilon > 0$ there exists a closed set D_ε such that $f_{|D_\varepsilon}$ is continuous and $\gamma(X - D_\varepsilon) < \varepsilon$.*

As a simple consequence of Theorems 1 and 2 we have

Theorem 3. *The regularity and the weak regularity of a small system $\{\mathcal{N}_n\}_{n \in N}$ on $\mathcal{S}$ are equivalent if and only if the $(\mathcal{N}_n)$—Lusin condition is equivalent to the $(\mathcal{N}_n)$—weak Lusin condition.*

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ON Θ-QUASI CONTINUOUS MULTIFUNCTIONS

1. Introduction

In 1963, Levine [8] has introduced the notion of semi-continuity between topological spaces. Since then several variations of continuity have been defined and investigated in the literature. Arya and Bhamini [1] introduced the notion of θ-semi-continuity as a generalization of semi-continuity. Recently, the second author [12] of the present paper has further investigated properties of θ-semi-continuity. In the present paper, we define and investigate upper (lower) θ-quasi continuous multifunctions. In Section 3, we shall obtain several characterizations of upper (lower) θ-quasi continuous multifunctions. In Section 4, we shall investigate several properties of such multifunctions.

2. Preliminaries

Throughout the present paper, X and Y always represent topological spaces. Let A be a subset of X. By $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, we denote the closure of A and the interior of A, respectively. The *θ-closure* [21] of A, denoted by $\mathrm{Cl}_\theta(A)$, is defined to be the set of all $x \in X$ such that $A \cap \mathrm{Cl}(U) \neq \emptyset$ for every open neighborhood U of x. The *θ-interior* [9] of A, denoted by $\mathrm{Int}_\theta(A)$, is defined to be the set of all $x \in A$ such that $\mathrm{Cl}(U) \subset A$ for some open neighborhood U of x. It is shown in [21] that $\mathrm{Cl}_\theta(A)$ is closed in X and that $\mathrm{Cl}(U) = \mathrm{Cl}_\theta(U)$ for each open set U of X. A subset A of X is said to be *semi-open* [8] if there exists an open set U such that $U \subset A \subset \mathrm{Cl}(U)$, or equivalently if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$. A subset A is said to be *regular open* (resp. *preopen* [10]) if $A = \mathrm{Int}(\mathrm{Cl}(A))$ (resp. $A \subset \mathrm{Int}(\mathrm{Cl}(A))$). The family of all semi-open (resp. regular open, preopen) sets of X is denoted by $\mathrm{SO}(X)$ (resp. $\mathrm{RO}(X), \mathrm{PO}(X)$). For each $x \in X$, the family of semi-open sets containing x is denoted by $\mathrm{SO}(X, x)$. The complement of a semi-open (resp. regular open) set is said to be *semi-closed* [2] (resp. *regular closed*). The family of all *semi-closed* (resp. *regular closed*) sets of X is denoted by $\mathrm{SC}(X)$ (resp.

RC(X)). The intersection of all semi-closed sets containing a subset A is called the *semi-closure* [2] of A and is denoted by sCl(A). The *semi-interior* of A, denoted by sInt(A), is defined by the union of all semi-open sets contained in A. The *semi θ-closure* of A and the *semi θ-interior* of A are respectively defined in [3] as follows:

$$\mathrm{sCl}_\theta(A) = \{x \in X \mid A \cap \mathrm{sCl}(U) \neq \emptyset \text{ for every } U \in \mathrm{SO}(X,x)\} \quad \text{and}$$
$$\mathrm{sInt}_\theta(A) = \{x \in A \mid \mathrm{sCl}(U) \subset A \text{ for some } U \in \mathrm{SO}(X,x)\}.$$

Throughout the present paper, $F : X \to Y$ (resp. $f : X \to Y$) represents a multifunction (resp. single valued function). For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a subset B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is

$$F^+(B) = \{x \in X \mid F(x) \subset B\} \quad \text{and} \quad F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}.$$

DEFINITION 1. A multifunction $F : X \to Y$ is said to be

(a) *upper weakly quasi continuous* [16] if for each $x \in X$, each open set U containing x and each open set V containing $F(x)$, there exists a nonempty open set G of X such that $G \subset U$ and $F(G) \subset \mathrm{Cl}(V)$;

(b) *lower weakly quasi continuous* [16] if for each $x \in X$, each open set U containing x and each open set V such that $F(x) \cap V \neq \emptyset$, there exists a nonempty open set G of X such that $G \subset U$ and $F(g) \cap \mathrm{Cl}(V) \neq \emptyset$ for every $g \in G$.

LEMMA 1 (Noiri and Popa [15]). *A multifunction $F : X \to Y$ is upper (resp. lower) weakly quasi continuous if and only if for each $x \in X$ and each open set V of Y such that $F(x) \subset V$ (resp. $F(x) \cap V \neq \emptyset$), there exists $U \in \mathrm{SO}(X,x)$ such that $F(U) \subset \mathrm{Cl}(V)$ (resp. $U \subset F^-(\mathrm{Cl}(V))$).*

DEFINITION 2. A multifunction $F : X \to Y$ is said to be

(a) *upper almost quasi continuous* [18] if for each $x \in X$, each open set U containing x and each open set V containing $F(x)$, there exists a nonempty open set G of X such that $G \subset U$ and $F(G) \subset \mathrm{sCl}(V)$;

(b) *lower almost quasi continuous* [18] if for each $x \in X$, each open set U containing x and each open set V such that $F(x) \cap V \neq \emptyset$, there exists a nonempty open set G of X such that $G \subset U$ and $F(g) \cap \mathrm{sCl}(V) \neq \emptyset$ for every $g \in G$.

LEMMA 2 (Popa and Noiri [18]). *The following are equivalent for a multifunction $F : X \to Y$:*

(1) *F is upper (resp. lower) almost quasi continuous;*
(2) *$F^+(V) \in \mathrm{SO}(X)$ (resp. $F^-(V) \in \mathrm{SO}(X)$ for every $V \in \mathrm{RO}(Y)$;*
(3) *$F^-(K) \in \mathrm{SC}(X)$ (resp. $F^+(K) \in \mathrm{SC}(X)$) for every $K \in \mathrm{RC}(Y)$;*

(4) *for each* $x \in X$ *and each open set* V *of* Y *such that* $F(x) \subset V$ (*resp.* $F(x) \cap V \neq \emptyset$), *there exists* $U \in \mathrm{SO}(X,x)$ *such that* $F(U) \subset \mathrm{sCl}(V) = \mathrm{Int}(\mathrm{Cl}(V))$ (*resp.* $U \subset F^-(\mathrm{sCl}(V)) = F^-(\mathrm{Int}(\mathrm{Cl}(V)))$).

3. Characterizations

DEFINITION 3. A multifunction $F : X \to Y$ is said to be

(a) *upper θ-quasi continuous* (briefly u.θ.q.c) for each point $x \in X$ and each open set V of Y containing $F(x)$, there exists $U \in \mathrm{SO}(X,x)$ such that $F(\mathrm{sCl}(U)) \subset \mathrm{Cl}(V)$;

(b) *lower θ-quasi continuous* (briefly l.θ.q.c.) if for each point $x \in X$ and each open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists $U \in \mathrm{SO}(X,x)$ such that $F(u) \cap \mathrm{Cl}(V) \neq \emptyset$ for every $u \in \mathrm{sCl}(U)$.

In this section we obtain several characterizations of upper (lower) θ-quasi continuous multifunctions.

THEOREM 1. *The following are equivalent for a multifunction* $F{:}X \to Y$:

(1) F *is u.θ.q.c.*;

(2) $\mathrm{sCl}_\theta(F^-(\mathrm{Int}(\mathrm{Cl}_\theta(B)))) \subset F^-(\mathrm{Cl}_\theta(B))$ *for every subset* B *of* Y;

(3) $\mathrm{sCl}_\theta(F^-(\mathrm{Int}(\mathrm{Cl}(V)))) \subset F^-(\mathrm{Cl}(V))$ *for every open set* V *of* Y;

(4) $\mathrm{sCl}_\theta(F^-(\mathrm{Int}(R))) \subset F^-(R)$ *for every* $R \in \mathrm{RC}(X)$;

(5) $F^+(V) \subset \mathrm{sInt}_\theta(F^+(\mathrm{Cl}(V)))$ *for every open set* V *of* Y;

(6) $\mathrm{sCl}_\theta(F^-(\mathrm{Int}(K))) \subset F^-(K)$ *for every closed set* K *of* Y;

(7) $\mathrm{sCl}_\theta(F^-(V)) \subset F^-(\mathrm{Cl}(V))$ *for every open set* V *of* Y.

Proof. (1)⇒(2): Let B be any subset of Y. Suppose that $x \notin F^-(\mathrm{Cl}_\theta(B))$. Then $x \in X - F^-(\mathrm{Cl}_\theta(B))$ and $F(x) \subset Y - \mathrm{Cl}_\theta(B)$. Since $\mathrm{Cl}_\theta(B)$ is closed in Y, there exists $U \in \mathrm{SO}(X,x)$ such that $F(\mathrm{sCl}(U)) \subset \mathrm{Cl}(Y - \mathrm{Cl}_\theta(B)) = Y - \mathrm{Int}(\mathrm{Cl}_\theta(B))$. Thus, we have $F(\mathrm{sCl}(U)) \cap \mathrm{Int}(\mathrm{Cl}_\theta(B)) = \emptyset$ and $\mathrm{sCl}(U) \cap F^-(\mathrm{Int}(\mathrm{Cl}_\theta(B))) = \emptyset$. This shows that $x \notin \mathrm{sCl}_\theta(F^-(\mathrm{Int}(\mathrm{Cl}_\theta(B))))$. Therefore, we obtain $\mathrm{sCl}_\theta(F^-(\mathrm{Int}(\mathrm{Cl}_\theta(B)))) \subset F^-(\mathrm{Cl}_\theta(B))$.

(2)⇒(3): This is obvious since $\mathrm{Cl}(V) = \mathrm{Cl}_\theta(V)$ for every open set V of Y.

(3)⇒(4): Let $R \in \mathrm{RC}(Y)$, then we have $\mathrm{sCl}_\theta(F^-(\mathrm{Int}(R))) = \mathrm{sCl}_\theta(F^-(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(R))))) \subset F^-(\mathrm{Cl}(\mathrm{Int}(R))) = F^-(R)$.

(4)⇒(5): Let V be any open set of Y. Then we have

$$X - \mathrm{sInt}_\theta(F^+(\mathrm{Cl}(V))) = \mathrm{sCl}_\theta(X - F^+(\mathrm{Cl}(V))) = \mathrm{sCl}_\theta(F^-(Y - \mathrm{Cl}(V))),$$

$$Y - \mathrm{Cl}(V) = \mathrm{Int}(Y - \mathrm{Cl}(V)) \subset \mathrm{Int}(Y - \mathrm{Int}(\mathrm{Cl}(V))), \quad \text{and}$$

$$Y - \mathrm{Int}(\mathrm{Cl}(V)) \in \mathrm{RC}(Y).$$

Therefore, we obtain

$$\mathrm{sCl}_\theta(F^-(\mathrm{Int}(Y - \mathrm{Int}(\mathrm{Cl}(V))))) \subset F^-(Y - \mathrm{Int}(\mathrm{Cl}(V))) = \\ = X - F^+(\mathrm{Int}(\mathrm{Cl}(V))) \subset X - F^+(V).$$

Consequently, we obtain $F^+(V) \subset \mathrm{sInt}_\theta(F^+(\mathrm{Cl}(V)))$.

(5)⇒(6): Let K be any closed set of Y. Then by (5) we have

$$X - F^-(K) = F^+(Y - K) \subset \mathrm{sInt}_\theta(F^+(\mathrm{Cl}(Y - K))) = \\ = \mathrm{sInt}_\theta(F^+(Y - \mathrm{Int}(K))) = \mathrm{sInt}_\theta(X - F^-(\mathrm{Int}(K))) = \\ = X - \mathrm{sCl}_\theta(F^-(\mathrm{Int}(K))).$$

Therefore, we obtain $\mathrm{sCl}_\theta(F^-(\mathrm{Int}(K))) \subset F^-(K)$.

(6)⇒(7): Let V be any open set of Y, then $\mathrm{Cl}(V)$ is closed and we have $\mathrm{sCl}_\theta(F^-(V)) \subset \mathrm{sCl}_\theta(F^-(\mathrm{Int}(\mathrm{Cl}(V)))) \subset F^-(\mathrm{Cl}(V))$.

(7)⇒(1): Let $x \in X$ and V be any open set of Y containing $F(x)$. Then $F(x) \cap \mathrm{Cl}(Y - \mathrm{Cl}(V)) = \emptyset$ and $x \notin F^-(\mathrm{Cl}(Y - \mathrm{Cl}(V)))$. It follows from (7) that $x \notin \mathrm{sCl}_\theta(F^-(Y - \mathrm{Cl}(V)))$. Then there exists $U \in \mathrm{SO}(X, x)$ such that $\mathrm{sCl}(U) \cap F^-(Y - \mathrm{Cl}(V) = \emptyset$; hence $F(\mathrm{sCl}(U)) \subset \mathrm{Cl}(V)$. This shows that F is u.θ.q.c.

Lemma 3. *If $F : X \to Y$ is l.θ.q.c., then for each $x \in X$ and each subset B of Y with $F(x) \cap \mathrm{Int}_\theta(B) \neq \emptyset$ there exists $U \in \mathrm{SO}(X, x)$ such that $\mathrm{sCl}(U) \subset F^-(B)$.*

Proof. Since $F(x) \cap \mathrm{Int}_\theta(B) \neq \emptyset$, there exists an open set V of Y such that $V \subset \mathrm{Cl}(V) \subset B$ and $F(x) \cap V \neq \emptyset$. Since F is l.θ.q.c., there exists $U \in \mathrm{SO}(X, x)$ such that $F(u) \cap \mathrm{Cl}(V) \neq \emptyset$ for every $u \in \mathrm{sCl}(U)$ and hence $\mathrm{sCl}(U) \subset F^-(B)$.

Theorem 2. *The following are equivalent for a multifunction $F : X \to Y$:*

(1) *F is l.θ.q.c.;*
(2) *$\mathrm{sCl}_\theta(F^+(B)) \subset F^+(\mathrm{Cl}_\theta(B))$ for every subset B of Y;*
(3) *$\mathrm{sCl}_\theta(F^+(V)) \subset F^+(\mathrm{Cl}(V))$ for every open set V of Y;*
(4) *$F^-(V) \subset \mathrm{sInt}_\theta(F^-(\mathrm{Cl}(V)))$ for every open set V of Y;*
(5) *$F(\mathrm{sCl}_\theta(A)) \subset \mathrm{Cl}_\theta(F(A))$ for every subset A of X;*
(6) *$\mathrm{sCl}_\theta(F^+(\mathrm{Int}(\mathrm{Cl}_\theta(B)))) \subset F^+(\mathrm{Cl}_\theta(B))$ for every subset B of Y;*
(7) *$\mathrm{sCl}_\theta(F^+(\mathrm{Int}(\mathrm{Cl}(V)))) \subset F^+(\mathrm{Cl}(V))$ for every open set V of Y;*
(8) *$\mathrm{sCl}_\theta(F^+(\mathrm{Int}(R))) \subset F^+(R)$ for every $R \in \mathrm{RC}(Y)$;*
(9) *$\mathrm{sCl}_\theta(F^+(\mathrm{Int}(K))) \subset F^+(K)$ for every closed set K of Y.*

Proof. (1)⇒(2): Let B be any subset of Y. Suppose that $x \notin F^+(\mathrm{Cl}_\theta(B))$. Then $x \in F^-(Y - \mathrm{Cl}_\theta(B)) = F^-(\mathrm{Int}_\theta(Y - B))$. Since F is l.θ.q.c., by Lemma 3 there exists $U \in \mathrm{SO}(X, x)$ such that $\mathrm{sCl}(U) \subset$

$F^-(Y-B) = X - F^+(B)$. Thus we have $\mathrm{sCl}(U) \cap F^+(B) = \emptyset$ and hence $x \notin \mathrm{sCl}_\theta(F^+(B))$.

(2)⇒(3): This is obvious since $\mathrm{Cl}(V) = \mathrm{Cl}_\theta(V)$ for every open set V of Y.

(3)⇒(4): Let V be any open set of Y. Then, we have

$$X - \mathrm{sInt}_\theta(F^-(\mathrm{Cl}(V))) = \mathrm{sCl}_\theta(X - F^-(\mathrm{Cl}(V))) = \mathrm{sCl}_\theta(F^+(Y - \mathrm{Cl}(V))) \subset$$
$$F^+(\mathrm{Cl}(Y - \mathrm{Cl}(V))) \subset F^+(\mathrm{Cl}(Y - V)) = F^+(Y - V) = X - F^-(V).$$

Therefore, we obtain $F^-(V) \subset \mathrm{sInt}_\theta(F^-(\mathrm{Cl}(V)))$.

(4)⇒(1): Let $x \in X$ and V be any open set such that $F(x) \cap V \neq \emptyset$. Then $x \in F^-(V) \subset \mathrm{sInt}_\theta(F^-(\mathrm{Cl}(V)))$. Therefore, there exists $U \in \mathrm{SO}(X, x)$ such that $\mathrm{sCl}(U) \subset F^-(\mathrm{Cl}(V))$; hence $F(u) \cap \mathrm{Cl}(V) \neq \emptyset$ for every $u \in \mathrm{sCl}(U)$. This shows that F is l.θ.q.c.

(2)⇒(5): Let A be any subset of X. By replacing B in (2) by $F(A)$, we have $\mathrm{sCl}_\theta(A) \subset \mathrm{sCl}_\theta(F^+(F(A))) \subset F^+(\mathrm{Cl}_\theta(F(A)))$. Thus we obtain $F(\mathrm{sCl}_\theta(A)) \subset \mathrm{Cl}_\theta(F(A))$.

(5)⇒(2): Let B be any subset of Y. Replacing A in (5) by $F^+(B)$, we have $F(\mathrm{sCl}_\theta(F^+(B))) \subset \mathrm{Cl}_\theta(F(F^+(B))) \subset \mathrm{Cl}_\theta(B)$. Thus we obtain $\mathrm{sCl}_\theta(F^+(B)) \subset F^+(\mathrm{Cl}_\theta(B))$.

(3)⇒(6): Let B be any subset of Y. Put $V = \mathrm{Int}(\mathrm{Cl}_\theta(B))$ in (3). Then, since $\mathrm{Cl}_\theta(B)$ is closed in Y, we have

$$\mathrm{sCl}_\theta(F^+(\mathrm{Int}(\mathrm{Cl}_\theta(B)))) \subset F^+(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}_\theta(B)))) \subset F^+(\mathrm{Cl}_\theta(B)).$$

(6)⇒(7): This is obvious since $\mathrm{Cl}(V) = \mathrm{Cl}_\theta(V)$ for any open set V of Y.

(7)⇒(8): If $R \in \mathrm{RC}(Y)$, then by (7) we have

$$\begin{aligned}\mathrm{sCl}_\theta(F^+(\mathrm{Int}(R))) &= \mathrm{sCl}_\theta(F^+(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(R)))))\\ &\subset F^+(\mathrm{Cl}(\mathrm{Int}(R))) = F^+(R).\end{aligned}$$

(8)⇒(9): Let K be any closed set of Y. Since $\mathrm{Cl}(\mathrm{Int}(K)) \in \mathrm{RC}(Y)$, we have

$$\begin{aligned}\mathrm{sCl}_\theta(F^+(\mathrm{Int}(K))) &= \mathrm{sCl}_\theta(F^+(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(K)))))\\ &\subset F^+(\mathrm{Cl}(\mathrm{Int}(K))) \subset F^+(K).\end{aligned}$$

(9)⇒(4): Let V be any open set of Y. Then $Y - V$ is closed in Y and by (9) we have $\mathrm{sCl}_\theta(F^+(\mathrm{Int}(Y - V))) \subset F^+(Y - V) = X - F^-(V)$. Moreover, we have $\mathrm{sCl}_\theta(F^+(\mathrm{Int}(Y - V))) = \mathrm{sCl}_\theta(F^+(Y - \mathrm{Cl}(V))) = \mathrm{sCl}_\theta(X - F^-(\mathrm{Cl}(V))) = X - \mathrm{sInt}_\theta(F^-(\mathrm{Cl}(V)))$. Therefore, we obtain $F^-(V) \subset \mathrm{sInt}_\theta(F^-(\mathrm{Cl}(V)))$.

COROLLARY 1 (Noiri [12]). *The following are equivalent for a function* $f : X \to Y$:

(1) *f is θ-semi-continuous;*

(2) $\mathrm{sCl}_\theta(f^{-1}(B)) \subset f^{-1}(\mathrm{Cl}_\theta(B))$ *for every subset* B *of* Y;
(3) $\mathrm{sCl}_\theta(f^{-1}(V)) \subset f^{-1}(\mathrm{Cl}(V))$ *for every open set* V *of* Y;
(4) $f^{-1}(V) \subset \mathrm{sInt}_\theta(f^{-1}(\mathrm{Cl}(V)))$ *for every open set* V *of* Y;
(5) $f(\mathrm{sCl}_\theta(A)) \subset \mathrm{Cl}_\theta(f(A))$ *for every subset* A *of* X.

For a multifunction $F : X \to Y$, the *graph multifunction* $G_F : X \to X \times Y$ is defined as follows: $G_F(x) = \{x\} \times F(x)$ for every $x \in X$.

LEMMA 4. (Noiri and Popa [14]). *The following hold for a multifunction* $F : X \to Y$:

(a) $G_F^+(A \times B) = A \cap F^+(B)$ *and* (b) $G_F^-(A \times B) = A \cap F^-(B)$ *for every subsets* $A \subset X$ *and* $B \subset Y$.

THEOREM 3. *Let* $F : X \to Y$ *be a multifunction such that* $F(x)$ *is compact for each* $x \in X$. *Then* F *is u.θ.q.c. if and only if* $G_F : X \to X \times Y$ *is u.θ.q.c..*

Proof. *Necessity*. Suppose that $F : X \to Y$ is u.θ.q.c. Let $x \in X$ and W be any open set of $X \times Y$ containing $G_F(x)$. For each $y \in F(x)$ there exist open sets $U(y) \subset Y$ and $V(y) \subset Y$ such that $(x, y) \in U(y) \times V(y) \subset W$. The family $\{V(y) \mid y \in F(x)\}$ is an open cover of $F(x)$ and there exists a finite number of points, says, $y_1, y_2, \ldots, y_n$ in $F(x)$ such that $F(x) \subset \bigcup\{V(y_i) \mid i = 1, 2, \ldots, n\}$. Set $U = \bigcap\{U(y_i) \mid i = 1, 2, \ldots, n\}$ and $V = \bigcup\{V(y_i) \mid i = 1, 2, \ldots, n\}$. Then U and V are open in X and Y, respectively, and $\{x\} \times F(x) \subset U \times V \subset W$. Since F is u.θ.q.c., there exists $U_0 \in \mathrm{SO}(X, x)$ such that $F(\mathrm{sCl}(U_0)) \subset \mathrm{Cl}(V)$. It follows from [11, Lemma 1] that $G = U \cap U_0 \in \mathrm{SO}(X, x)$. By Lemma 4 we have

$$\begin{aligned}\mathrm{sCl}(G) = \mathrm{sCl}(U \cap U_0) &\subset \mathrm{sCl}(U) \cap \mathrm{sCl}(U_0) \subset \mathrm{Cl}(U) \cap F^+(\mathrm{Cl}(V)) \\ &= G_F^+(\mathrm{Cl}(U) \times \mathrm{Cl}(V)) = G_F^+(\mathrm{Cl}(U \times V)) \subset G_F^+(\mathrm{Cl}(W)).\end{aligned}$$

Thus $G_F(\mathrm{sCl}(G)) \subset \mathrm{Cl}(W)$. This shows that G_F is u.θ.q.c.

Sufficiency. Suppose that $G_F : X \to X \times Y$ is u.θ.q.c. Let $x \in X$ and V be any open set of Y containing $F(x)$. Since $X \times V$ is open in $X \times Y$ and $G_F(x) \subset X \times Y$, there exists $U \in \mathrm{SO}(X, x)$ such that $G_F(\mathrm{sCl}(U)) \subset \mathrm{Cl}(X \times V) = X \times \mathrm{Cl}(V)$. Therefore by Lemma 4 we obtain $\mathrm{sCl}(U) \subset G_F^+(X \times \mathrm{Cl}(V)) = F^+(\mathrm{Cl}(V))$ and hence $F(\mathrm{sCl}(U)) \subset \mathrm{Cl}(V)$. This shows that F is u.θ.q.c.

THEOREM 4. *A multifunction* $F : X \to Y$ *is l.θ.q.c. if and only if* $G_F : X \to X \times Y$ *is l.θ.q.c.*

Proof. *Necessity*. Suppose that F is l.θ.q.c. Let $x \in X$ and W be any open set of $X \times Y$ such that $G_F(x) \cap W \neq \emptyset$. There exists $y \in F(x)$ such that $(x, y) \in W$ and hence we have $(x, y) \in U \times V \subset W$ for some open

sets $U \subset X$ and $V \subset Y$. Since F is l.θ.q.c. and $y \in F(x) \cap V$, there exists $U_0 \in \mathrm{SO}(X,x)$ such that $\mathrm{sCl}(U_0) \subset F^-(\mathrm{Cl}(V))$. It follows from [11, Lemma 1] that $G = U \cap U_0 \in \mathrm{SO}(X,x)$. By Lemma 4, we have

$$\begin{aligned}\mathrm{sCl}(G) &= \mathrm{sCl}(U \cap U_0) \subset \mathrm{sCl}(U) \cap \mathrm{sCl}(U_0) \subset \mathrm{Cl}(U) \cap F^-(\mathrm{Cl}(V)) \\ &= G_F^-(\mathrm{Cl}(U) \times \mathrm{Cl}(V)) = G_F^-(\mathrm{Cl}(U \times V)) \subset G_F^-(\mathrm{Cl}(W)).\end{aligned}$$

Therefore, $G_F(u) \cap \mathrm{Cl}(W) \neq \emptyset$ for every $u \in \mathrm{sCl}(G)$. This shows that G_F is l.θ.q.c.

Sufficiency. Suppose that G_F is l.θ.q.c. Let $x \in X$ and V be an open set in Y such that $F(x) \cap V \neq \emptyset$. Then $X \times V$ is open in $X \times Y$ and $G_F(x) \cap (X \times V) = (\{x\} \times F(x)) \cap (X \times V) = \{x\} \times (F(x) \cap V) \neq \emptyset$. There exists $U \in \mathrm{SO}(X,x)$ such that $G_F(u) \cap \mathrm{Cl}(X \times V) \neq \emptyset$ for each $u \in \mathrm{sCl}(U)$. By Lemma 4, we obtain $\mathrm{sCl}(U) \subset G_F^-(\mathrm{Cl}(X \times V)) = F^-(\mathrm{Cl}(V))$. This shows that F is l.θ.q.c.

Corollary 2 (Noiri [12]). *A function $f : X \to Y$ is θ-semi-continuous if and only if the graph function $g : X \to X \times Y$ is θ-semi-continuous.*

For a multifunction $F : X \to Y$, a multifunction $\mathrm{sCl}\,F : X \to Y$ is defined in [17] as follows: $(\mathrm{sCl}\,F)(x) = \mathrm{sCl}(F(x))$ for each $x \in X$.

Lemma 5. (Noiri and Popa [14]). *Let $F : X \to Y$ be a multifunction. Then $(\mathrm{sCl}\,F)^-(V) = F^-(V)$ for every $V \in \mathrm{SO}(Y)$.*

Theorem 5. *A multifunction $F : X \to Y$ is l.θ.q.c. if and only if $\mathrm{sCl}\,F : X \to Y$ is l.θ.q.c.*

Proof. *Necessity.* Suppose that F is l.θ.q.c. Let $x \in X$ and V be any open set of Y such that $(\mathrm{sCl}\,F)(x) \cap V \neq \emptyset$. By Lemma 5, we have $F(x) \cap V \neq \emptyset$. Since F is l.θ.q.c., there exists $U \in \mathrm{SO}(X,x)$ such that $F(u) \cap \mathrm{Cl}(V) \neq \emptyset$ for every $u \in \mathrm{sCl}(U)$. Since $\mathrm{Cl}(V) \in \mathrm{SO}(Y)$, by Lemma 5 we have $\mathrm{sCl}(U) \subset F^-(\mathrm{Cl}(V)) = (\mathrm{sCl}\,F)^-(\mathrm{Cl}(V))$ and hence $(\mathrm{sCl}\,F)(u) \cap \mathrm{Cl}(V) \neq \emptyset$ for any $u \in \mathrm{sCl}(U)$. This shows that $\mathrm{sCl}\,F$ is l.θ.q.c.

Sufficiency. Suppose that $\mathrm{sCl}\,F$ is l.θ.q.c. Let $x \in X$ and V be any open set of Y such that $F(x) \cap V \neq \emptyset$. Then $\mathrm{sCl}(F(x)) \cap V \neq \emptyset$ and there exists $U \in \mathrm{SO}(X,x)$ such that $(\mathrm{sCl}\,F)(u) \cap \mathrm{Cl}(V) \neq \emptyset$ for every $u \in \mathrm{sCl}(U)$. Since $\mathrm{Cl}(V) \in \mathrm{SO}(Y)$, by Lemma 5 $\mathrm{sCl}(U) \subset (\mathrm{sCl}\,F)^-(\mathrm{Cl}(V)) = F^-(\mathrm{Cl}(V))$ and hence $F(u) \cap \mathrm{Cl}(V) \neq \emptyset$ for every $u \in \mathrm{sCl}(U)$. Therefore, F is l.θ.q.c.

4. Some results

First, we obtain some sufficient conditions for an upper (resp. lower) weakly quasi continuous multifunction to be u.θ.q.c. (resp. l.θ.q.c.).

Theorem 6. *If a multifunction $F : X \to Y$ is upper weakly quasi continuous and lower almost quasi continuous, then F is u.θ.q.c.*

P r o o f. Let $x \in X$ and V be any open set of Y containing $F(x)$. Then, by Lemma 1, there exists $U \in \mathrm{SO}(X,x)$ such that $F(U) \subset \mathrm{Cl}(V)$ and hence $U \subset F^+(\mathrm{Cl}(V))$. Since $\mathrm{Cl}(V) \in \mathrm{RC}(Y)$, by Lemma 2 we have $F^+(\mathrm{Cl}(V)) \in \mathrm{SC}(X)$. Therefore, we obtain $\mathrm{sCl}(U) \subset F^+(\mathrm{Cl}(V))$ and hence $F(\mathrm{sCl}(U)) \subset \mathrm{Cl}(V)$. This shows that F is u.θ.q.c.

THEOREM 7. *If a multifunction $F : X \to Y$ is lower weakly quasi continuous and upper almost quasi continuous, then F is l.θ.q.c.*

P r o o f. Let $x \in X$ and V be any open set such that $F(x) \cap V \neq \emptyset$. Since F is lower weakly quasi continuous, by Lemma 1 there exists $U \in \mathrm{SO}(X,x)$ such that $U \subset F^-(\mathrm{Cl}(V))$. Since $\mathrm{Cl}(V) \in \mathrm{RC}(Y)$, by Lemma 2 $F^-(\mathrm{Cl}(V)) \in \mathrm{SC}(X)$ and hence $\mathrm{sCl}(U) \subset F^-(\mathrm{Cl}(V))$. This implies that $F(u) \cap \mathrm{Cl}(V) \neq \emptyset$ for every $u \in \mathrm{sCl}(U)$. Thus F is l.θ.q.c.

COROLLARY 3. (Arya and Bhamini [1]). *Every almost semi-continuous function is θ-semi-continuous.*

DEFINITION 4. A topological space X is said to be *semi-regular* [4] if for each semi-closed set A and each point $x \notin A$, there exist disjoint semi-open sets U and V such that $x \in U$ and $A \subset V$.

THEOREM 8. *If a multifunction $F : X \to Y$ is upper weakly quasi continuous and X is semi-regular, then F is u.θ.q.c.*

P r o o f. Let $x \in X$ and V be an open set of Y containing $F(x)$. Then, by Lemma 1, there exists $G \in \mathrm{SO}(X,x)$ such that $F(G) \subset \mathrm{Cl}(V)$. By [4, Theorem 2.1], there exists $U \in \mathrm{SO}(X,x)$ such that $x \in U \subset \mathrm{sCl}(U) \subset G$. Therefore, we obtain $F(\mathrm{sCl}(U)) \subset \mathrm{Cl}(V)$ and hence F is u.θ.q.c.

THEOREM 9. *If a multifunction $F : X \to Y$ is lower weakly quasi continuous and X is semi-regular, then F is l.θ.q.c.*

P r o o f. The proof is similar to that of Theorem 8.

DEFINITION 5. A subset of X is said to be *α-almost regular* [7] if for any point $a \in A$ and any $U \in \mathrm{RO}(X)$ containing a, there exists an open set G such that $a \in G \subset \mathrm{Cl}(G) \subset U$.

DEFINITION 6. A subset A of X is said to be *α-nearly paracompact* [6] if every cover of A by regular open sets of X has an X-open X-locally finite refinement which covers A.

LEMMA 6 (Kovačević [7]). *If A is an α-almost regular α-nearly paracompact subset of X and U is a regular open neighborhood of A, then there exists an open neighborhood G of A such that $A \subset G \subset \mathrm{Cl}(G) \subset U$.*

THEOREM 10. *If a multifunction $F : Y \to Y$ is upper weakly quasi continuous and $F(x)$ is α-almost regular α-nearly paracompact in Y for each $x \in X$, then F is upper almost quasi continuous.*

Proof. Let V be any regular open set of Y and $F(x) \subset V$. Since $F(x)$ is α-almost regular α-nearly paracompact, by Lemma 6 there exists an open set W of Y such that $F(x) \subset W \subset \mathrm{Cl}(W) \subset V$. Since F is upper weakly quasi continuous, by Lemma 1 there exists $U \in \mathrm{SO}(X, x)$ such that $F(U) \subset \mathrm{Cl}(W) \subset V$. Therefore, we have $x \in U \subset F^{+}(V)$ and hence $F^{+}(V) \in \mathrm{SO}(X)$. It follows from Lemma 2 that F is upper almost quasi continuous.

COROLLARY 4. *If $F : X \to Y$ is an u.θ.q.c. multifunction and $F(x)$ is α-almost regular α-nearly paracompact in Y for each $x \in X$, then F is upper almost quasi continuous.*

LEMMA 7 (Noiri and Ahmad [13]). *Let A and X_0 be subsets of X. Then the following hold:*

(a) *If $A \in \mathrm{SO}(X)$ and $X_0 \in \mathrm{PO}(X)$, then $A \cap (X_0) \in \mathrm{SO}(X_0)$.*

(b) *If $A \subset X_0 \in \mathrm{PO}(X)$, then $X_0 \cap \mathrm{sCl}(A) = \mathrm{sCl}_{X_0}(A)$, where $\mathrm{sCl}_{X_0}(A)$ denotes the semi-closure of A in the subspace X_0 of X.*

THEOREM 11. *If a multifunction $F : X \to Y$ is u.θ.q.c. and $X_0 \in \mathrm{PO}(X)$, then the restriction $F \mid X_0 : X_0 \to Y$ is u.θ.q.c.*

Proof. Let $x \in X_0$ and V be any open set of Y containing $F(x)$. There exists $U \in \mathrm{SO}(X, x)$ such that $F(\mathrm{sCl}(U)) \subset \mathrm{Cl}(V)$. Since $X_0 \in \mathrm{PO}(X)$, by Lemma 7 we have $x \in U \cap X_0 \in \mathrm{SO}(X_0)$ and $\mathrm{sCl}_{X_0}(U \cap X_0) = X_0 \cap \mathrm{sCl}(U \cap X_0) \subset X_0 \cap \mathrm{sCl}(U)$. Therefore, we obtain

$$\begin{aligned}(F \mid X_0)(\mathrm{sCl}_{X_0}(U \cap X_0)) &\subset (F \mid X_0)(X_0 \cap \mathrm{sCl}(U)) \\ &= F(X_0 \cap \mathrm{sCl}(U)) \subset \mathrm{Cl}(V).\end{aligned}$$

This shows that $F \mid X_0$ is u.θ.q.c.

THEOREM 12. *If a multifunction $F : X \to Y$ is l.θ.q.c. and $X_0 \in \mathrm{PO}(X)$, then the restriction $F \mid X_0 : X_0 \to Y$ is l.θ.q.c.*

Proof. Let $x \in X_0$ and V be any open set of Y such that $F(x) \cap V \neq \emptyset$. There exists $U \in \mathrm{SO}(X, x)$ such that $F(u) \cap \mathrm{Cl}(V) \neq \emptyset$ for each $u \in \mathrm{sCl}(U)$. Since $X_0 \in \mathrm{PO}(X)$, we have $x \in U \cap X_0 \in \mathrm{SO}(X_0)$ and $\mathrm{sCl}_{X_0}(U \cap X_0) \subset X_0 \cap \mathrm{sCl}(U)$. For any $u \in \mathrm{sCl}_{X_0}(U \cap X_0)$, $u \in \mathrm{sCl}(U) \cap X_0$ and $\emptyset \neq F(u) \cap \mathrm{Cl}(V) = (F \mid X_0)(u) \cap \mathrm{Cl}(V)$. Therefore, we have $(F \mid X_0)(u) \cap \mathrm{Cl}(V) \neq \emptyset$ for every $u \in \mathrm{sCl}_{X_0}(U \cap X_0)$. This shows that $F \mid X_0$ is l.θ.q.c.

Corollary 5 (Noiri [12]). *If a function $f : X \to Y$ is a θ-semi-continuous and $X_0 \in \mathrm{PO}(X)$, then the restriction $f \mid X_0 : X_0 \to Y$ is θ-semi-continuous.*

Definition 7. A topological space X is said to be *quasi H-closed* [19] (resp. *S-closed* [20], *s-closed* [3]) if for every open (resp. semi-open) cover $\{U_\alpha \mid \alpha \in \nabla\}$ of X, there exists a finite subset ∇_0 of ∇ such that $X = \bigcup\{\mathrm{Cl}(U_\alpha) \mid \alpha \in \nabla_0\}$ (resp. $X = \bigcup\{\mathrm{Cl}(U_\alpha) \mid \alpha \in \nabla_0\}, X = \bigcup\{\mathrm{sCl}(U_\alpha) \mid \alpha \in \nabla_0\}$).

It is well known that s-closedness implies S-closedness and S-closedness implies quasi H-closedness but none of these implications is reversible.

Theorem 13. *Let $F : X \to Y$ be a surjective multifunction and $F(x)$ compact for each $x \in X$. If F is u.θ.q.c. and X is s-closed, then Y is quasi H-closed.*

Proof. Let $\{V_\alpha \mid \alpha \in \nabla\}$ be any open cover of Y. Let x be any point of X. Since $F(x)$ is compact, there exists a finite subset $\nabla(x)$ of ∇ such that $F(x) \subset \bigcup\{V_\alpha \mid \alpha \in \nabla(x)\}$. Put $V(x) = \bigcup\{V_\alpha \mid \alpha \in \nabla(x)\}$, then $F(x) \subset V(x)$ and $V(x)$ is open in Y. Since F is u.θ.q.c., there exists $U(x) \in \mathrm{SO}(X, x)$ such that $F(\mathrm{sCl}(U(x))) \subset \mathrm{Cl}(V(x))$. The family $\{U(x) \mid x \in X\}$ is a semi-open cover of X. Since X is s-closed, there exists a finite number of points, says, $x_1, x_2, \ldots, x_n$ in X such that $X = \bigcup\{\mathrm{sCl}(U(x_i)) \mid i = 1, 2, \ldots, n\}$. Since F is surjective, we obtain

$$Y = F(X) \subset \bigcup_{i=1}^{n} F(\mathrm{sCl}(U(x_i))) \subset \bigcup_{i=1}^{n} \mathrm{Cl}(V(x_i)) = \bigcup_{i=1}^{n} \bigcup_{\alpha \in \nabla(x_i)} \mathrm{Cl}(\nabla_\alpha).$$

This shows that Y is quasi H-closed.

Theorem 14. *Let $F : X \to Y$ be a surjective multifunction and $F(x)$ compact for each $x \in X$. If F is upper weakly quasi continuous and lower almost quasi continuous and X is S-closed, then Y is quasi H-closed.*

Proof. Let $\{V_\alpha \mid \alpha \in \nabla\}$ be any open cover of Y. For each $x \in X$, $F(x)$ is compact and there exists a finite subset $\nabla(x)$ of ∇ such that $F(x) \subset \bigcup\{V_\alpha \mid \alpha \in \nabla(x)\}$. Now, put $V(x) = \bigcup\{V_\alpha \mid \alpha \in \nabla(x)\}$, then $V(x)$ is open in Y and $F(x) \subset V(x)$. It follows from Theorem 6 that F is u.θ.q.c. By Theorem 1, we have $x \in F^+(V(x)) \subset \mathrm{sInt}_\theta(F^+(\mathrm{Cl}(V(x)))) \in \mathrm{SO}(X)$ and hence $\{\mathrm{sInt}_\theta(F^+(\mathrm{Cl}(V(x)))) \mid x \in X\}$ is a semi-open cover of X. Since X is S-closed, there exists a finite number of points, says, $x_1, x_2, \ldots, x_n$ in X such that

$$X = \bigcup_{i=1}^{n} \mathrm{Cl}(\mathrm{sInt}_\theta(F^+(\mathrm{Cl}(V(x_i))))) =$$

$$= \bigcup_{i=1}^{n} \mathrm{Cl}(F^+(\mathrm{Cl}(V(x_i)))) = \mathrm{Cl}\left(\bigcup_{i=1}^{n} F^+(\mathrm{Cl}(V(x_i)))\right).$$

It follows from [5, Theorem 2.4] that

$$X = \mathrm{sCl}\left(\bigcup_{i=1}^{n} F^+(\mathrm{Cl}(V(x_i)))\right) \subset \mathrm{sCl}\left(F^+\left(\bigcup_{i=1}^{n} \mathrm{Cl}(V(x_i))\right)\right)$$

$$= \mathrm{sCl}\left(F^+\left(\bigcup_{i=1}^{n} V(x_i)\right)\right).$$

Since $\mathrm{Cl}(\bigcup\{V(x_i) \mid 1 \leq i \leq n\})$ is regular closed in Y, by Lemma 2 $F^+(\mathrm{Cl}(\bigcup\{V(x_i) \mid 1 \leq i \leq n\}))$ is semi-closed in X. Therefore, we have

$$Y = F(X) = F(F^+(\mathrm{Cl}(\bigcup_{i=1}^{n} V(x_i)))) \subset \mathrm{Cl}(\bigcup_{i=1}^{n} V(x_i)) =$$

$$= \bigcup_{i=1}^{n} \mathrm{Cl}(V(x_i)) = \bigcup_{i=1}^{n} \bigcup_{\alpha \in V(x_i)} \mathrm{Cl}(V_\alpha).$$

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ON GAUGE–NATURAL OPERATORS OF CURVATURE TYPE ON PAIRS OF CONNECTIONS

Recently, Kolář has determined all gauge–natural operators for any Lie group G of curvature type transforming every principal connection into modified curvature operator, [3]. In the case the structure group is the general linear group $GL(m,\mathbf{R})$ in an arbitrary dimension m he has obtained two parameter family of all $GL(m,\mathbf{R})$–natural operators of curvature type consisting of a curvature of connection and a contracted curvature of connection and generated by linear adjoint invariant maps of the Lie algebra $gl(m,\mathbf{R})$ into itself of the form id and $A \mapsto \operatorname{tr} A \cdot \mathrm{id}$, [4].

Using a general method by Kolář, [3], [4], [5], we determine all gauge–natural operators for any Lie group G defined on the bundle $\mathbf{Q} \oplus \mathbf{Q}$ of pairs of principal connections with values in $L \otimes \otimes^2 T^* B$. We deduce that all such operators form a system generated by two modified curvatures of both principal connections and by values on the difference of these connections of some map induced by bilinear adjoint invariant map of the Lie algebra $\mathfrak{g}$ of G. In the case the structure group is the general linear group $GL(m,\mathbf{R})$ in an arbitrary dimension m we obtain 10–parameter family of all $GL(m,\mathbf{R})$–natural operators of curvature type consisting 4–parameter system generated by curvatures and contracted curvatures both connections and 6–parameter system generated by values on the difference of these connections of some bilinear and adjoint invariant maps of the Lie algebra $gl(m,\mathbf{R})$.

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1. Let $\mathcal{M}f_n$ be a category of n–dimensional manifolds and their local diffeomorphisms. Let $\mathcal{FM}$ be a category of fibred manifolds and denote by B the base functor. Fix a Lie group G and define a category $\mathcal{P}_n(G)$, whose objects are principal G–bundles and whose morphisms are the morphisms

of principal G–bundles $f : P \to \overline{P}$ with the base map $Bf : BP \to B\overline{P}$ in $\mathcal{M}f_n$.

DEFINITION 1. A gauge–natural bundle over n–manifolds is a functor $F : P_n(G) \to FM$ such that

1. every principal bundle $p : P \to BP$ in $P_n(G)$ is transformed by F into fibred manifold $q : FP \to BP$ over the same base BP
2. every morphism $f : P \to \overline{P}$ in $P_n(G)$ is transformed by F into morphism $Ff : FP \to F\overline{P}$ over Bf in FM
3. for every open subset $U \subset BP$, the inclusion $\imath : p^{-1}(U) \to P$ is transformed into the inclusion $F\imath : q^{-1}(U) \to FP$.

Let F and E be two G–natural bundle over n–manifolds.

DEFINITION 2. A gauge–natural operator $A : F \to E$ is a system of operators $A_P : C^\infty FP \to C^\infty EP$ for every object P in $P_n(G)$ transforming every section $s \in C^\infty FP$ into section $A_P s \in C^\infty EP$ such that

1. $A_{\overline{P}}(Ff \circ s \circ Bf^{-1}) = Ef \circ A_P s \circ Bf^{-1}$ for every isomorphism $f : P \to \overline{P}$ in $P_n(G)$
2. $A_{P^{-1}(U)}(s|U) = (A_P s)|U$ for every open subset $U \subset BP$
3. A_P transforms every smoothly parametrized family of sections into smoothly parametrized family of sections.

A gauge–natural bundle F over n–manifolds is said to be of order r, if for any two morphisms $f, h : P \to \overline{P}$ in $P_n(G)$ the condition $j^r_y f = j^r_y h$ at some point $y \in P_x$ of the fibre of P over $x \in BP$ implies $Ff|_{F_xP} = Fh|_{F_xP}$.

Let $W^r P$ be a space of all r–jets $j^r_{(0,e)}\varphi$, where $\varphi : \mathbf{R}^n \times G \to P$ is a morphism in $P_n(G)$. The space $W^r P$ is a principal bundle over BP with structure group $W^r_n G$, which is the group of all r–jets $j^r_{(0,e)}\psi$ of morphisms $\psi : \mathbf{R}^n \times G \to \mathbf{R}^n \times G$ in $P_n(G)$ satisfying $B\psi(0,e) = 0$. Every morphism $f : P \to \overline{P}$ in $P_n(G)$ is extended into a principal bundle morphism $W^r f : W^r P \to W^r \overline{P}$ defined by the jet composition $W^r f(j^r_{(0,e)}\varphi) = j^r_{(0,e)}(f \circ \varphi)$. Every smooth left action of $W^r_n G$ on a manifold S determines r–th order G–natural bundle over n–manifolds as a functor transforming every object P in $P_n(G)$ into the fibre bundle associated to $W^r P$ with standard fibre S and every morphism f in $P_n(G)$ into $(W^r f, \mathrm{id}_S)$.

Every r–th order gauge–natural bundle is a fibre bundle associated to bundle W^r. The k–th jet prolongation $J^k F$ of a gauge–natural bundle F of order r is a gauge–natural bundle of order $(k + r)$.

According to a general theory, [1], [2], [3], [4], [5], there is a canonical bijection between the k–th order G–natural operators $A : F \to E$ and the

$W_n^s G$-equivariant maps of standard fibres $A : J_0^k F(\mathbf{R}^n \times G) \to E_0(\mathbf{R}^n \times G)$, where s is maximum of the orders $J^k F$ and E.

2. The connection bundle $\mathbf{Q}P \to BP$ of P can be defined as the factor space $\mathbf{Q}P = J^1P/G$ over BP. Clearly, the connection bundle $\mathbf{Q} : P_n(G) \to \mathcal{F}M$, $\mathbf{Q} : P \mapsto \mathbf{Q}P$, is a gauge–natural bundle of the order 1.

Given a connection Γ on $P = \mathbf{R}^n \times G$, its value $\Gamma(0,e)$ is a 1-jet $j_0^1\gamma$ of a section $\gamma : \mathbf{R}^n \to \mathbf{R}^n \times G$, which is identified with a map $\gamma : \mathbf{R}^n \to G$, where $\gamma(0) = e$ is a unit of G.

The standard fibre of $\mathbf{Q}$ is of the form $\mathbf{Q}_0(\mathbf{R}^n \times G) = J_0^1(\mathbf{R}^n, G)_e = \mathbf{g} \otimes \mathbf{R}^{n*}$. Fix a basis e_p of the Lie algebra $\mathbf{g}$ of G, we have the coordinate expression of an element Γ of $\mathbf{g} \otimes \mathbf{R}^{n*}$ in the form $\Gamma = \Gamma_i^p e_p\, dx^i$.

Consider an isomorphism $\Phi : \mathbf{R}^n \times G \to \mathbf{R}^n \times G$ in $P_n(G)$ of the form

(2.1) $$\overline{x} = f(x), \quad \overline{y} = \varphi(x) \cdot y$$

where $f : \mathbf{R}^n \to \mathbf{R}^n$ is a local diffeomorphism satisfying $f(0) = 0$ and $\varphi : \mathbf{R}^n \to G$ is a smooth map and the dot denotes the multiplication in G.

Then 2–jet $j_0^2\Phi \in W_n^2 G$ has coordinates

(2.2) $$\begin{aligned} &a = \varphi(0) \in G \\ &(a_i^p, a_{ij}^p) = j_0^2(\varphi(x) \cdot a^{-1}) \in \mathbf{g} \otimes \mathbf{R}^{n*} \times \mathbf{g} \otimes S^2\mathbf{R}^{n*} \\ &(a_j^i, a_{jk}^i) = j_0^2 f \in G_n^2. \end{aligned}$$

Isomorphism Φ transforms a section γ generating $\Gamma(0,e)$ into a section $\varphi(f^{-1}(x)) \cdot \gamma(f^{-1}(x))$ generating image $\Phi(\Gamma(0,e))$ as the 1–jet of the section

(2.3) $$\Phi(\Gamma(0,e)) = j_0^1[\varphi(f^{-1}(x)) \cdot \gamma(f^{-1}(x)) \cdot a^{-1}].$$

Let $A_q^p(a)$ be a coordinate expression of the adjoint representation of G in Lie algebra $\mathbf{g}$. Let tilda denote the inverse matrix. Then (2.3) gives the action of $W_n^1 G$ on the standard fibre $\mathbf{Q}_0 = \mathbf{g} \otimes \mathbf{R}^{n*}$ in the form

(2.4) $$\overline{\Gamma}_i^p = A_q^p(a)(\Gamma_j^q + a_j^q)\widetilde{a}_i^j.$$

Let $\Gamma_{ij}^p = \frac{\partial \Gamma_i^p}{\partial x^j}$ be the induced coordinates on $J_0^1\mathbf{Q} = \mathbf{g} \otimes \mathbf{R}^{n*} \times \mathbf{g} \otimes S^2\mathbf{R}^{n*}$. Using a general prolongation procedure, [3], we deduce from (2.4) the action of $W_n^2 G$ on $J_0^1\mathbf{Q}$ in the form (2.4) and

(2.5) $$\begin{aligned} \overline{\Gamma}_{ij}^p = {} & A_q^p(a)\Gamma_{kl}^q \widetilde{a}_i^k \widetilde{a}_j^l + A_q^p(a) a_{kl}^q \widetilde{a}_i^k \widetilde{a}_j^l + \\ & + B_{qr}^p(a)\Gamma_k^q a_l^r \widetilde{a}_i^k \widetilde{a}_j^l + E_{qr}^p(a) a_k^q a_l^r \widetilde{a}_i^k \widetilde{a}_j^l + \\ & + A_q^p(a)(\Gamma_k^q + a_k^q)\widetilde{a}_{ij}^k , \end{aligned}$$

where B_{qr}^p and E_{qr}^p are some functions on G.

The curvature of a principal connection can be considered as a section $C_P\Gamma : BP \to LP \otimes \Lambda^2 T^*BP$, where LP is an accociated bundle to P with

standard fibre $\mathbf{g}$ with respect to adjoint action of Lie group G. The curvature operator $C : \mathbf{Q} \to L \otimes \Lambda^2 T^* B$ is a gauge–natural operator of the first order because of geometric definition of the curvature.

In order to obtain a coordinate expression of curvature, we write the equation of a connection Γ on $P = \mathbf{R}^n \times G$ in the form

$$\omega^p = \Gamma_i^p(x)\, dx^i, \tag{2.6}$$

where ω^p are the Maurer–Cartan forms on G with respect to the basis e_p of Lie algebra $\mathbf{g}$. Then, components of a connection form are

$$\eta^p = \omega^p - \Gamma_i^p(x)\, dx^i. \tag{2.7}$$

Using a structure equation of the connection Γ in the form

$$d\eta^p = C^p_{qr}\eta^q \wedge \eta^r + R^p_{ij} dx^i \wedge dx^j \tag{2.8}$$

and Maurer–Cartan equation

$$d\omega^p = C^p_{qr}\omega^q \wedge \omega^r, \tag{2.9}$$

we obtain a coordinate expression of curvature in the form

$$R^p_{ij} = \Gamma^p_{ij} - \Gamma^p_{ji} + C^p_{qr}\Gamma^q_i\Gamma^r_j, \tag{2.10}$$

where C^p_{qr} are a structure constant of G.

Let $Z \subset \mathrm{Lin}(\mathbf{g},\mathbf{g})$ be the subspace of all linear maps commuting with the adjoint action of G. Since every $z \in Z$ is an equivariant linear map of the standard fibre $\mathbf{g}$ of the vector bundle LP, it induces a morphism $z_P : LP \to LP$. Hence, we can construct a modified curvature operator of the curvature operator C_P in the form $C(z)_P = (z_P \otimes \Lambda^2 T^* \mathrm{id}_{BP}) \circ C_P$.

We shall need some new essential relations concerning the function B^p_{qr} on G appearing in (2.5), which we will obtain in the detailed proof of the following theorem developed by I. Kolář in [3].

THEOREM 1. *All gauge–natural operators $\mathbf{Q} \to L \otimes \otimes^2 T^* B$ are the modified curvature operators*

$$C(z) = (z \otimes \Lambda^2 T^* \mathrm{id}_B) \circ C. \tag{2.11}$$

for all $z \in Z$ of the curvature operator C.

P r o o f. I. The first order gauge–natural operators $A : \mathbf{Q} \to L \otimes \otimes^2 T^* B$ are in bijection with $W^2_n G$ equivariant maps of standard fibres $A : J^1_0 \mathbf{Q} \to \mathbf{g} \otimes \otimes^2 \mathbf{R}^{n*}$. The group $W^2_n G$ acts on the standard fibre $J^1_0 \mathbf{Q}$ by formulas (2.4) and (2.5). On $\mathbf{g} \otimes \otimes^2 \mathbf{R}^{n*}$ we have the canonical coordinates y^p_{ij} and we have the action of $W^2_n G$ in the form

$$\overline{y}^p_{ij} = A^p_q(a) y^q_{kl} \widetilde{a}^k_i \widetilde{a}^l_j. \tag{2.12}$$

The equivariancy of A with respect to homotheties in $G^1_n : a = e$, $\widetilde{a}^i_j = k\delta^i_j$, $a^i_{jk} = 0$, $a^p_i = 0$, $a^p_{ij} = 0$, gives a homogeneity condition

$$k^2 f^p_{ij}(\Gamma^p_i, \Gamma^p_{ij}) = f^p_{ij}(k \cdot \Gamma^p_i, k^2 \Gamma^p_{ij}). \tag{2.13}$$

By the homogeneous function theorem, [5], we deduce that f^p_{ij} are linear in Γ^p_{ij} and quadratic in Γ^p_i. Using invariant tensor theorem for G^1_n, [5], we obtain f^p_{ij} in the form

$$f^p_{ij} = b^p_q \Gamma^q_{ij} + d^p_q \Gamma^q_{ji} + k^p_{qr} \Gamma^q_i \Gamma^r_j \tag{2.14}$$

with real coefficients.

Considering equivariancy of f^p_{ij} in the form (2.14) with respect to the subgroup in $W^2_n G : a = e$, $\widetilde{a}^i_j = \delta^i_j$ and a^i_{jk}, a^p_i, a^p_{ij} are arbitrary, we get conditions

$$\begin{gathered} b^p_q + d^p_q = 0 \\ b^p_q B^q_{rs}(e) + k^p_{rs} = 0, \qquad d^p_q B^q_{rs}(e) + k^p_{sr} = 0. \end{gathered} \tag{2.15}$$

From this, we obtain the following relations

$$d^p_q = -b^p_q, \quad k^p_{sr} = -k^p_{rs}, \quad B^q_{sr}(e) = -B^q_{rs}(e), \quad k^p_{rs} = -b^p_q B^q_{rs}(e). \tag{2.16}$$

If we put $b^p_q = \delta^p_q$ into f^p_{ij} in (2.14) and if we use uniqueness of the curvature operator, we can put

$$B^q_{rs}(e) = -C^q_{rs}, \tag{2.17}$$

where C^q_{rs} are the constant structure of G.

The equivariancy of f^p_{ij} in (2,14) with respect to the canonical inclusion $\imath(G)$ into $W^2_n G : a \in G$ is arbitrary and $\widetilde{a}^i_j = \delta^i_j$, $a^i_{jk} = 0$, $a^p_i = 0$, $a^p_{ij} = 0$, gives the relation

$$A^p_q(a) b^q_r R^r_{ij} = b^p_q A^q_r(a) R^r_{ij}. \tag{2.18}$$

This means that b^p_q commutes with adjoint action.

II. The r–th order gauge–natural operators $\mathbf{Q} \to L \otimes \otimes^2 T^* B$ correspond to $W^{r+1}_n G$ equivariant maps of standard fibres $J^r_0 \mathbf{Q} \to \mathbf{g} \otimes \otimes^2 \mathbf{R}^{n*}$.

Let $\Gamma^p_{i\alpha}$ be the induced coordinates on $J^r_0 \mathbf{Q}$, where α is a multiindex of range n with $|\alpha| \leq r$.

Equivariancy $f^p_{ij}(\Gamma^q_{k\alpha})$ with respect to homotheties in G^1_n gives a homogeneity condition

$$k^2 f^p_{ij}(\Gamma^q_{k\alpha}) = f^p_{ij}(k^{1+|\alpha|} \Gamma^q_{k\alpha}). \tag{2.19}$$

By homogeneous function theorem f^p_{ij} is independent on $\Gamma^q_{k\alpha}$ for $|\alpha| \geq 2$ and is linear in Γ^p_{ij} and quadratic in Γ^p_i. Hence, the r–th order operators are reduced to the case I for every $r \geq 2$. Moreover, every gauge–natural

operator defined on the connection bundle has a finite order. This proves Theorem 1.

3. Consider a pair of principal connections Γ and Δ on the principal bundle $P = \mathbf{R}^n \times G$ as a section of the bundle $\mathbf{Q} \oplus \mathbf{Q}$. We are going to determine all gauge–natural operators $\mathbf{Q} \oplus \mathbf{Q} \to L \otimes \otimes^2 T^* B$.

Let $Y \subset \mathrm{Billin}(\mathbf{g} \times \mathbf{g}, \mathbf{g})$ be the subspace of all bilinear and adjoint equivariant maps with respect to adjoint action of G.

For every $y \in Y$ we define a bilinear map $\overline{y} : \mathbf{g} \otimes \mathbf{R}^{n*} \times \mathbf{g} \otimes \mathbf{R}^{n*} \to \mathbf{g} \otimes \otimes^2 \mathbf{R}^{n*}$ by formula

$$\overline{y}(A_1 \otimes X_1, A_2 \otimes X_2) = y(A_1, A_2) \otimes X_1 \otimes X_2. \tag{3.1}$$

Now, we prove the main

THEOREM 2. *All gauge–natural operators* $\mathbf{Q} \oplus \mathbf{Q} \to L \otimes \otimes^2 T^* B$ *form the following system*

$$(\Gamma, \Delta) \mapsto C(z)(\Gamma) + C(w)(\Delta) + \overline{y}(D, D), \tag{3.2}$$

where $C(z)(\Gamma)$ *and* $C(w)(\Delta)$ *are modified curvature operators for all* $z, w \in Z$ *and* $\overline{y}(D, D)$ *are the values on the difference of connections* $D = \Gamma - \Delta$ *for all* $y \in Y$.

Proof. I. The first order gauge–natural operators $\mathbf{Q} \oplus \mathbf{Q} \to L \otimes \otimes^2 T^* B$ correspond bijectively to the $W^2_n G$ equivariant maps of the standard fibres $J^1_0(\mathbf{Q} \oplus \mathbf{Q}) \to \mathbf{g} \otimes \otimes^2 \mathbf{R}^{n*}$. The group $W^2_n G$ acts on the standard fibre $J^1_0(\mathbf{Q} \oplus \mathbf{Q})$ by formulas (2.4) and (2.5) and by formulas

$$\begin{aligned} \overline{\Delta}^p_i &= A^p_q(a)(\Delta^q_j + a^q_j)\widetilde{a}^j_i \\ \overline{\Delta}^p_{ij} &= A^p_q(a)\Delta^q_{kl}\widetilde{a}^k_i\widetilde{a}^l_j + A^p_q(a)a^q_{kl}\widetilde{a}^k_i\widetilde{a}^l_j + \\ &\quad + B^p_{qr}(a)\Delta^q_k a^r_l \widetilde{a}^k_i\widetilde{a}^l_j + E^p_{qr}(a)a^q_k a^r_l \widetilde{a}^k_i\widetilde{a}^l_j + \\ &\quad + A^p_q(a)(\Delta^q_k + a^q_k)\widetilde{a}^k_{ij}. \end{aligned} \tag{3.3}$$

The action of $W^2_n G$ on $\mathbf{g} \otimes \otimes^2 \mathbf{R}^{n*}$ is of the form (2.12).

Any map of standard fibres $A : J^1_0(\mathbf{Q} \oplus \mathbf{Q}) \to \mathbf{g} \otimes \otimes^2 \mathbf{R}^{n*}$ in coordinates is of the form

$$y^p_{ij} = f^p_{ij}(\Gamma^q_k, \Gamma^q_{kl}, \Delta^q_k, \Delta^q_{kl}). \tag{3.4}$$

The equivariancy of f^p_{ij} with respect to homotheties in G^1_n : $a = e$, $\widetilde{a}^i_j = k\delta^i_j$, $a^i_{jk} = 0$, $a^p_i = 0$, $a^p_{ij} = 0$, gives a homogeneity condition

$$k^2 f^p_{ij}(\Gamma^p_i, \Gamma^p_{ij}, \Delta^p_i, \Delta^p_{ij}) = f^p_{ij}(k\Gamma^p_i, k^2\Gamma^p_{ij}, k\Delta^p_i, k^2\Delta^p_{ij}). \tag{3.5}$$

By the homogeneous function theorem, [5], we deduce that f^p_{ij} are linear in Γ^p_{ij}, Δ^p_{ij} and bilinear in Γ^p_i, Δ^p_i and quadratic in Γ^p_i, Δ^p_i. Using invariant

tensor theorem for G^1_n, we obtain f^p_{ij} in the form

$$f^p_{ij} = b^p_q \Gamma^q_{ij} + c^p_q \Gamma^q_{ji} + d^p_q \Delta^q_{ij} + e^p_q \Delta^q_{ji} + k^p_{qr} \Gamma^q_i \Delta^r_j + + l^p_{qr} \Delta^q_i \Gamma^r_j + m^p_{qr} \Gamma^q_i \Gamma^r_j + n^p_{qr} \Delta^q_i \Delta^r_j \tag{3.6}$$

with real coefficients.

Considering equivariancy of f^p_{ij} with respect to the subgroup in $W^2_n G$: $a = e$, $a^i_j = \delta^i_j$ and a^i_{jk}, a^p_i, a^p_{ij} are arbitrary, we get the following relations

$$\begin{aligned} c^p_q &= -b^p_q, \quad e^p_q = -d^p_q \\ k^p_{rs} &= -m^p_{rs} - b^p_q B^q_{rs}(e) \\ l^p_{rs} &= -m^p_{rs} + b^p_q B^q_{sr}(e) \\ k^p_{rs} &= -n^p_{rs} + d^p_q B^q_{sr}(e) \\ l^p_{rs} &= -n^p_{rs} - d^p_q B^q_{rs}(e). \end{aligned} \tag{3.7}$$

Using the relation (2.16), $B^q_{rs}(e) = -B^q_{sr}(e)$, we get $k^p_{rs} = l^p_{rs}$. If we put

$$h^p_{rs} = -k^p_{rs}, \quad h^p_{rs} = -l^p_{rs} \tag{3.8}$$

and take into account the relation (2.17) $B^q_{rs}(e) = -C^q_{rs}$, we obtain following relations

$$\begin{aligned} m^p_{rs} &= h^p_{rs} + b^p_q C^q_{rs} \\ n^p_{rs} &= h^p_{rs} + d^p_q C^q_{rs}. \end{aligned} \tag{3.9}$$

Finally, f^p_{ij} is of the form

$$f^p_{ij} = b^p_q \overset{\Gamma}{R}{}^q_{ij} + d^p_q \overset{\Delta}{R}{}^q_{ij} + h^p_{qr} D^q_i \cdot D^r_j \tag{3.10}$$

where we denote

$$\begin{aligned} \overset{\Gamma}{R}{}^q_{ij} &= \Gamma^q_{ij} - \Gamma^q_{ji} + C^q_{rs} \Gamma^r_i \Gamma^s_j, \\ \overset{\Delta}{R}{}^q_{ij} &= \Delta^q_{ij} - \Delta^q_{ji} + C^q_{rs} \Delta^r_i \Delta^s_j, \\ D^q_i &= \Gamma^q_i - \Delta^q_i. \end{aligned} \tag{3.11}$$

Considering equivariancy of the map (3.10), f^p_{ij}, with respect to the canonical inclusion $\imath(G)$ into $W^2_n G : a \in G$ is arbitrary and $a^i_j = \delta^i_j$, $a^i_{jk} = 0$, $a^p_i = 0$, $a^p_{ij} = 0$, we obtain the relation:

$$\begin{aligned} & A^p_q(a) b^q_r \overset{\Gamma}{R}{}^r_{ij} + A^p_q(a) d^q_r \overset{\Delta}{R}{}^r_{ij} + A^p_q(a) h^q_{rs} D^r_i D^s_j = \\ & = b^p_q A^q_r(a) \overset{\Gamma}{R}{}^r_{ij} + d^p_q A^q_r(a) \overset{\Delta}{R}{}^r_{ij} + h^p_{qr} A^q_s(a) A^r_t(a) D^s_i D^t_j. \end{aligned} \tag{3.12}$$

This means that b^p_q and d^p_q defines linear maps of $\mathbf{g}$ into itself commuting with adjoint action $A^p_q(a)$ and h^p_{qr} define a bilinear map $\mathbf{g}\otimes\mathbf{R}^{n*}\times\mathbf{g}\otimes\mathbf{R}^{n*}\to\mathbf{g}\otimes\otimes^2\mathbf{R}^{n*}$ determined by some linear adjoint equivariant map $\mathbf{g}\times\mathbf{g}\to\mathbf{g}$.

II. The r–th order gauge–natural operators $\mathbf{Q}\oplus\mathbf{Q}\to L\otimes\otimes^2T^*B$ correspond bijectively to the W^{r+1}_nG equivariant maps of the standard fibres $J^r_0(\mathbf{Q}\oplus\mathbf{Q})\to\mathbf{g}\otimes\otimes^2\mathbf{R}^{n*}$. Let $(\Gamma^p_{i\alpha},\Delta^p_{i\alpha})$ be induced coordinates on $J^r_0(\mathbf{Q}\oplus\mathbf{Q})$, where α is a multiindex of range n with $|\alpha|\leq r$.

Equivariancy of $y^p_{ij}=f^p_{ij}(\Gamma^p_{i\alpha},\Delta^p_{i\alpha})$ with respect to homotheties in G^1_n gives a homogeneity condition

$$k^2f^p_{ij}(\Gamma^p_{i\alpha},\Delta^p_{i\alpha})=f^p_{ij}(k^{1+|\alpha|}\Gamma^p_{i\alpha},k^{1+|\alpha|}\Delta^p_{i\alpha}).\tag{3.13}$$

By homogeneous function theorem f^p_{ij} is independent on $\Gamma^p_{i\alpha}$, $\Delta^p_{i\alpha}$ for $|\alpha|\geq 2$ and is linear in Γ^p_{ij}, Δ^p_{ij} and is bilinear in Γ^p_i, Δ^p_i and is quadratic in Γ^p_i, Δ^p_i. Hence, the r–th order operators are reduced to the case I. Moreover, since every gauge–natural operator $\mathbf{Q}\to L\otimes\otimes^2T^*B$ defined on $\mathbf{Q}$ has a finite order so gauge–natural operator $\mathbf{Q}\oplus\mathbf{Q}\to L\otimes\otimes^2T^*B$ defined on $\mathbf{Q}\oplus\mathbf{Q}$ has a finite order, too. This proves our theorem.

4. Consider a principal bundle $P=\mathbf{R}^n\times G^1_m$ with the general linear group $G^1_m=GL(m,\mathbf{R})$ in an arbitrary dimension m as a structure group. Let x^i,x^p_q be the canonical coordinates on the product bundle $\mathbf{R}^n\times G^1_m$, where $i,j,\ldots=1,\ldots,n$ and $p,q,\ldots=1,\ldots,m$.

The equation of connection Γ on $\mathbf{R}^n\times G^1_m$ are

$$dx^p_q=\Gamma^p_{ri}(x)x^r_q\,dx^i,\tag{4.1}$$

where Γ^p_{ri} are smooth functions defined on $\mathbf{R}^n$.

The curvature of the connection Γ on $P=\mathbf{R}^n\times G^1_m$ is a section $C\Gamma:\mathbf{R}^n\to L(\mathbf{R}^n\times G^1_m)\otimes\Lambda^2T^*\mathbf{R}^n$ of the form in local coordinates:

$$C\Gamma=(\Gamma^p_{qij}+\Gamma^p_{ri}\Gamma^r_{qj})\frac{\partial}{\partial x^p_q}\otimes dx^i\wedge dx^j.\tag{4.2}$$

In the case of the structure group G^1_m, all linear adjoint equivariant maps of $\mathbf{g}l(m,\mathbf{R})=\mathbf{R}^m\otimes\mathbf{R}^{m*}$ into itself form the 2–parameter family generated by id and $A\mapsto(\operatorname{tr}A)\,\mathrm{id}$. This gives the 2–parameter family of G^1_m–natural operators $\mathbf{Q}\to L\otimes\otimes^2T^*B$ generated by the curvature operator C and by a contracted curvature operator $\overline{C}=(\tau\otimes\Lambda^2\,\mathrm{id})\circ C$, where τ is a linear map of $L(\mathbf{R}^n\times G^1_m)$ into itself defined by the linear adjoint equivariant map of the standard fibre $\mathbf{g}$ into itself of the form $A\mapsto(\operatorname{tr}A)\,\mathrm{id}$.

In a local coordinates on $P=\mathbf{R}^n\times G^1_m$ the contracted curvature of the connection Γ is of the form

$$\overline{C}\Gamma=\delta^p_q(\Gamma^r_{rij})\frac{\partial}{\partial x^p_q}\otimes dx^i\wedge dx^j.\tag{4.3}$$

We will use the following

LEMMA 3. *All bilinear and adjoint invariant maps* $\mathbf{gl}(m,\mathbf{R})\times\mathbf{gl}(m,\mathbf{R})\to\mathbf{gl}(m,\mathbf{R})$ *form the 6–parameter family*

$$w^p_q = k_1 y^p_s z^s_q + k_2 y^s_q z^p_s + k_3 y^p_q z^s_s + k_4 y^s_s z^p_q + \\ + k_5 \delta^p_q y^r_r z^s_s + k_6 \delta^p_q y^r_s z^s_r \tag{4.4}$$

for any real parameters $k_1,\dots,k_6$.

Proof. Using the invariant tensor theorem for G^1_m, [5], we obtain any bilinear and adjoint invariant map $\mathbf{R}^m\otimes\mathbf{R}^{m*}\times\mathbf{R}^m\otimes\mathbf{R}^{m*}\to\mathbf{R}^m\otimes\mathbf{R}^{m*}$ in the form

$$w^p_q = k_1\delta^t_q\delta^p_s\delta^r_u y^s_r z^u_t + k_2\delta^r_q\delta^t_s\delta^p_u y^s_r z^u_t + \\ + k_3\delta^r_q\delta^p_s\delta^t_u y^s_r z^u_t + k_4\delta^t_q\delta^r_s\delta^p_u y^s_r z^u_t + \\ + k_5\delta^p_q\delta^t_s\delta^r_u y^s_r z^u_t + k_6\delta^p_q\delta^r_s\delta^t_u y^s_r z^u_t \tag{4.5}$$

with any real parameter $k_1,\dots,k_6$.

Consider a pair of connections Γ and Δ on $P=\mathbf{R}^n\times G^1_m$ as a section $(\Gamma,\Delta):\mathbf{R}^n\to\mathbf{Q}(\mathbf{R}^n\times G^1_m)\oplus\mathbf{Q}(\mathbf{R}^n\times G^1_m)$ with equations (4.1) and $dx^p_q=\Delta^p_{qi}(x)dx^i$.

We denote by

$$D^p_{qi}=\Gamma^p_{qi}-\Delta^p_{qi} \tag{4.6}$$

coordinates of difference of connections Γ and Δ and by

$$\overset{\Gamma}{R}{}^p_{qij}=\Gamma^p_{qij}-\Gamma^p_{qji}+\Gamma^p_{ri}\Gamma^r_{qj}-\Gamma^p_{rj}\Gamma^r_{qi}, \tag{4.7}$$

$$\overset{\Gamma}{R}{}^r_{rij}=\Gamma^r_{rij}-\Gamma^r_{rji} \tag{4.8}$$

coordinates of the curvature $C\Gamma$ and the contracted curvature $\overline{C}\Gamma$ of the connection Γ, respectively.

Immediately from Theorem 2 and Lemma 3 we obtain the following

COROLLARY 4. *All* $GL(m,\mathbf{R})$*–natural operators* $\mathbf{Q}\oplus\mathbf{Q}\to L\otimes\otimes^2T^*B$ *form the 10–parameter family*

$$(\Gamma,\Delta)\mapsto a_1\overset{\Gamma}{R}{}^p_{qij}+a_2\delta^p_q\overset{\Gamma}{R}{}^r_{rij}+a_3\overset{\Delta}{R}{}^p_{qij}+a_4\delta^p_q\overset{\Delta}{R}{}^r_{rij}+ \\ a_5D^p_{ri}D^r_{qj}+a_6D^p_{rj}D^r_{qi}+a_7D^r_{ri}D^p_{qj}+ \\ a_8D^p_{qi}D^r_{rj}+a_9\delta^p_qD^r_{si}D^s_{rj}+a_{10}\delta^p_qD^r_{ri}D^s_{sj} \tag{4.9}$$

with any real parameters $a_1,\dots,a_{10}$.

This (4.9) the 10–parameter family of $GL(m,\mathbf{R})$-natural operators $\mathbf{Q}\oplus\mathbf{Q}\to L\otimes\otimes^2T^*B$ may be also obtained by direct calculation.

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STRONG APPROXIMATION IN GENERALIZED HÖLDER NORMS

In this paper we extend some results on approximation of 2π-periodic functions, given in [3]–[5], to the case of strong approximation by some means of Fourier series. Similarly as in [3], we present approximation results in Hölder norms based on general modulus-type functions.

1. Notations

1.1. Let $C = C_{2\pi}$ be the space of 2π-periodic real-valued functions continuous on $R = (-\infty, +\infty)$ with the norm

(1) $$\|f\|_C := \max_x |f(x)|.$$

Let Ω be the set of modulus-type functions, i.e. Ω is the set of all functions Ω satisfying the following conditions:

a) ω is defined and continuous on $\langle 0, +\infty)$,
b) ω is increasing and $\omega(0) = 0$,
c) $\omega(h)h^{-1}$ is decreasing for $h > 0$.

For a given $\omega \in \Omega$ we define the class H^ω of all functions $f \in C$ for which

(2) $$\|f\|_\omega := \sup_{h>0} \frac{\|\Delta_h f\|_C}{\omega(h)} < +\infty,$$

where

(3) $$\Delta_h f(x) = f(x+h) - f(x).$$

In H^ω we define the norm

(4) $$\|f\|_{H^\omega} := \|f\|_C + \|f\|_\omega.$$

It is known that H^ω ($\omega \in \Omega$) with the norm (4) is a Banach space. H^ω is called generalized Hölder space. If $\omega(h) = h^\alpha$, $0 < \alpha \leq 1$, then H^ω is the

classical Hölder space.

1.2. Similarly as in [3]–[5] we define subspace $\widetilde{H}^\omega \subset H^\omega$, $\omega \in \Omega$, as follows

$$\widetilde{H}^\omega := \left\{ f \in H^\omega : \lim_{h \to 0_+} \frac{\|\Delta_h f\|_C}{\omega(h)} = 0 \right\}$$

with the norm $\|\cdot\|_{\tilde{H}^\omega}$ defined by (4).

If $\omega, \mu \in \Omega$ and $q(h) = \frac{\omega(h)}{\mu(h)}$, $h > 0$, is a non-decreasing function, then

$$H^\omega \subset H^\mu \quad \text{and} \quad \widetilde{H}^\omega \subset \widetilde{H}^\mu. \tag{5}$$

1.3. Denote, as usual, by $E_n(f;C)$, $n \in N = \{0, 1, \ldots\}$, the best approximation of function $f \in C$ by trigonometric polynomials of order $\leq n$ in the sense of C. It is known ([6], [8]) that if $f \in H^\omega$, $\omega \in \Omega$, then

$$E_n(f;C) \leq 3\omega\left(\frac{1}{n+1}\right)\|f\|_\omega, \quad n \in N. \tag{6}$$

If $f \in \widetilde{H}^\omega$, $\omega \in \Omega$, then

$$E_n(f;C) = o\left(\omega\left(\frac{1}{n+1}\right)\right) \quad \text{as } n \to \infty. \tag{7}$$

1.4. For a given $f \in C$ let

$$\frac{a_o}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx) \tag{8}$$

be its Fourier series. Let $S_n(x; f)$, $n \in N$, $x \in R$, be the n-th partial sum of (8).

In this paper we shall consider the strong approximation of function f belonging to the spaces H^ω by some means of Fourier series (8) in generalized Hölder norms (4).

Let Q be a set of real numbers, $r_o \notin Q$ be a accumulation point of Q and let $T = \{t_k(r)\}_{k=0}^\infty$ be a sequence of real-valued functions defined on Q and such that:

1° $t_k(r) \geq 0$ for all $r \in Q$ and $k \in N$,

2° $\lim\limits_{\substack{r \to r_o \\ r \in Q}} t_k(r) = 0$ for every fixed $k \in N$,

3° the series $\sum_{k=1}^\infty t_k(r) log(k+1)$ is convergent on Q,

4° $\sum_{n=0}^\infty 2^n T_{\bar{n}, 2\bar{n}}(r) \leq M_1$, for all $r \in Q$, where $\overline{n} = 2^n - 1$, $M_1 =$ const. > 0 and

$$T_{p,q}(r) := \max_{p \leq k \leq q} t_k(r), \quad p, q \in N. \tag{9}$$

We define the strong T-means of Fourier series (8) of $f \in C$:

$$L(r,x;f,T) := \sum_{k=0}^{\infty} t_k(r)|S_k(x;f) - f(x)| \tag{10}$$

for $x \in R$ and $r \in Q$. It is obvious that if $f \in C$, then for every fixed $r \in Q$ the function $L(r, \cdot\,; f, T)$ belongs also to C.

The purpose of this note is to estimate the generalized Hölder norms of $L(r, \cdot\,; f, T)$ for $f \in H^\omega$ and $f \in \widetilde{H}^\omega$.

By $M_k(\cdot)$, $k = 1, 2, \ldots$, we shall denote suitable positive constants depending only on indicated parameters.

2. Auxiliary results

Let $P = \{p_k\}_{k=0}^{\infty}$ be a non-negative and bounded sequence,

$$P_{n,m} := \max_{n \le k \le m} p_k, \quad n, m \in N, \tag{11}$$

and let $\{v_n\}$ be a monotone non-decreasing sequence of integers such that $0 \le v_n \le n$ and $n \le \lambda v_n$ for $n \in N$, where $\lambda = \text{const.} \ge 1$. For a given $f \in C$ we define two functions as follows:

$$U_{n,v_n}(x;f,P) := \frac{1}{v_n + 1} \sum_{k=n-v_n}^{n} p_k |S_k(x;f)| \tag{12}$$

and

$$W_{n,v_n}(x;f,P) := \frac{1}{v_n + 1} \sum_{k=n-v_n}^{n} p_k |S_k(x;f) - f(x)|, \tag{13}$$

$x \in R$. Clearly the functions (12) and (13) belong to C also.

Using the Leindler results given in [1] and [2] ([2], Th. 1.11), we immediately obtain

LEMMA 1. *If $f \in C$, then we have*

$$\|U_{n,v_n}(\cdot\,;f,P)\|_C \le M_1(\lambda) P_{n-v_n,n} \|f\|_C \tag{14}$$

and

$$\|W_{n,v_n}(\cdot\,;f,P)\|_C \le M_2(\lambda) P_{n-v_n,n} E_{n-v_n}(f;C) \tag{15}$$

for all $n \in N$.

Using Lemma 1, we shall prove two lemmas.

LEMMA 2. *If $f \in H^\omega$, $\omega \in \Omega$, then*

$$\|W_{n,v_n}(\cdot\,;f,P)\|_\omega \le M_4(\lambda) P_{n-v_n,n} \|f\|_\omega$$

for all $n \in N$, which proves that $W_{n,v_n}(\cdot\,;f,P) \in H^\omega$.

Proof. By (1)–(3) and (11)–(13), we have

$$\|W_{n,v_n}(\cdot;f,P)\|_\omega = \sup_{h>0}\{\|\Delta_h W_{n,v_n}(\cdot;f,P)\|_C/\omega(h)\}$$

and

$$\begin{aligned}|\Delta_h W_{n,v_n}(x;f,P)| &= \left|\frac{1}{v_n+1}\sum_{k=n-v_n}^{n} p_k\Delta_h|S_k(x;f)-f(x)|\right| \\ &\le \frac{1}{v_n+1}\sum_{k=n-v_n}^{n} p_k|\Delta_h S_k(x;f)-\Delta_h f(x)| \\ &= \frac{1}{v_n+1}\sum_{k=n-v_n}^{n} p_k|S_k(x;\Delta_h f)-\Delta_h f(x)| \\ &\le U_{n,v_n}(x;\Delta_h f,P)+P_{n-v_n,n}|\Delta_h f(x)|\end{aligned}$$

for all $x\in R$. Hence and by (14) we get

$$\|\Delta_h W_{n,v_n}(\cdot;f,P)\|_C \le (M_1(\lambda)+1)P_{n-v_n,n}\|\Delta_h f\|_C, \tag{16}$$

which implies

$$\|W_{n,v_n}(.;f,P)\|_\omega \le (M_1(\lambda)+1)P_{n-v_n,n}\|f\|_\omega.$$

Thus the proof is completed.

LEMMA 3. *Suppose that $f\in H^\omega$, $\omega\in\Omega$, and $\mu\in\Omega$ is a function such that $q(h)=\frac{\omega(h)}{\mu(h)}$ is non-decreasing for $h>0$. Then we have*

$$\|W_{n,v_n}(\cdot;f,P)\|_\mu \le M_5(\lambda)P_{n-v_n,n}\,q\Big(\frac{1}{n-v_n+1}\Big)\|f\|_\omega$$

for all $n\in N$.

Proof. From Lemma 2 and (5) it follows that $W_{n,v_n}(\cdot;f,P)\in H^\mu$. Hence and by (1)–(3) we have

$$\begin{aligned}\|W_{n,v_n}(\cdot;f,P)\|_\mu &= \sup_{h>0}(\|\Delta_h W_{n,v_n}(\cdot;f,P)\|_C/\mu(h)) \\ &\le \|W_{n,v_n}(\cdot;f,P)\|_{\mu,1}+\|W_{n,v_n}(\cdot;f,P)\|_{\mu,2},\end{aligned}$$

where

$$\|W_{n,v_n}(\cdot;f,P)\|_{\mu,1} := \sup_{h\in A_n}(\|\Delta_h W_{n,v_n}(\cdot;f,P)\|_C/\mu(h)),$$

$$\|W_{n,v_n}(\cdot;f,P)\|_{\mu,2} := \sup_{h\in B_n}(\|\Delta_h W_{n,v_n}(\cdot;f,P)\|_C/\mu(h)),$$

$$A_n=\{h: h>1/(n-v_n+1)\},\quad B_n=\{h: 0<h\le 1/(n-v_n+1)\},$$

By (3) we have

$$\|\Delta_h W_{n,v_n}(\,\cdot\,; f, P)\|_C \le 2\|W_{n,v_n}(\,\cdot\,; f, P)\|_C,$$

which, by (15) and (6), gives

$$\begin{aligned}\|W_{n,v_n}(\,\cdot\,; f, P)\|_{\mu,1} &\le 2\left(\mu\left(\frac{1}{n-v_n+1}\right)\right)^{-1} \|W_{n,v_n}(\,\cdot\,; f, P)\|_C \\ &\le 2M_2(\lambda)\left(\mu\left(\frac{1}{n-v_n+1}\right)\right)^{-1} P_{n-v_n,n} E_{n-v_n}(f; C) \\ &\le 6M_2(\lambda) P_{n-v_n,n}\, q\left(\frac{1}{n-v_n+1}\right)\|f\|_\omega.\end{aligned}$$

Using (16), we get

$$\begin{aligned}\|W_{n,v_n}(\,\cdot\,; f, P)\|_{\mu,2} &\le (M_1(\lambda)+1) P_{n-v_n,n} \sup_{h\in B_n} \frac{\|\Delta_h f\|_C}{\mu(h)} \\ &\le (M_1(\lambda)+1) P_{n-v_n,n}\, q\left(\frac{1}{n-v_n+1}\right)\|f\|_\omega.\end{aligned}$$

Summing up, we obtain our result.

Applying (4), (15), (6) and Lemma 3, we obtain the following

LEMMA 4. *Under the assumptions of Lemma 3 we have*

$$\|W_{n,v_n}(\,\cdot\,; f, P)\|_{H^\mu} \le M_5(\lambda, \mu) P_{n-v_n,n}\, q\left(\frac{1}{n-v_n+1}\right)\|f\|_\omega$$

for all $n \in N$.

Arguing as in the proofs of Lemmas 2–4 and using (1)–(7) we obtain

LEMMA 5. *If* $f \in \widetilde{H}^\omega$, $\omega \in \Omega$, *then the function* $W_{n,v_n}(\,\cdot\,; f, P)$, $n \in N$, $0 \le v_n \le n$, *belongs to* $\widetilde{H}^\omega$ *also.*

LEMMA 6. *Suppose that* $f \in \widetilde{H}^\omega$, $\omega \in \Omega$ *and* $\mu \in \Omega$ *is such that* $q(h) = \frac{\omega(h)}{\mu(h)}$ *is a non-decreasing function for* $h > 0$. *Then, if* $P_{n-v_n,n} > 0$ *and* $n - v_n \to \infty$, *we have*

$$\|W_{n,v_n}(\,\cdot\,; f, P)\|_{\tilde{H}^\mu} = o\left(P_{n,v_n,n}\, q\left(\frac{1}{n-v_n+1}\right)\right) \quad \text{as } n \to \infty.$$

In particular, if $P_{n,2n} > 0$, then

$$\|W_{2n,n}(\,\cdot\,; f, P)\|_{\tilde{H}^\mu} = o\left(P_{n,2n}\, q\left(\frac{1}{n+1}\right)\right) \quad \text{as } n \to \infty.$$

3. General theorems

In this part we shall give four theorems on the strong means $L(r,\cdot;f,T)$ of Fourier series of $f \in C$. We observe that if T is a sequence having the properties 1°–4° and $f \in C$, then formula (10) can be written in the form

$$(17)\qquad L(r,x;f,T) = \sum_{n=0}^{\infty}\sum_{k=\overline{n}}^{2\overline{n}} t_k(r)|S_k(x;f)-f(x)| = \sum_{n=0}^{\infty} 2^n W_{2\overline{n},\overline{n}}(r,x;f,T),$$

for $x \in R$ and $r \in Q$, where $\overline{n} = 2^n - 1$ and $W_{2\overline{n},\overline{n}}(r,x;f,T)$ is defined by (13) with $P \equiv T$.

From (4) and (17) we get

$$\Delta_h L(r,x;f,T) = \sum_{n=0}^{\infty} 2^n \Delta_h W_{2\overline{n},\overline{n}}(r,x;f,T),$$

which, by (16), (9) and 4°, gives that

$$(18)\qquad \begin{aligned}\|\Delta_h L(r,\cdot;f,T)\|_C &\le \sum_{n=0}^{\infty} 2^n \|\Delta_h W_{2\overline{n},\overline{n}}(r,\cdot;f,T)\|_C \\ &\le M_6 \|\Delta_h f\|_C \sum_{n=0}^{\infty} 2^n T_{\overline{n},2\overline{n}}(r) \\ &\le M_7 \|\Delta_h f\|_C, \quad r \in Q,\ h > 0.\end{aligned}$$

From (18) we immediately obtain

THEOREM 1. *If $f \in H^{\omega}$, $\omega \in \Omega$, then*

$$\|L(r,\cdot;f,T)\|_{\omega} \le M_8 \|f\|_{\omega}, \quad r \in Q,$$

which proves that $L(r,\cdot;f,T)\| \in H^{\omega}$ for every fixed $r \in Q$.

THEOREM 2. *If $f \in \widetilde{H}^{\omega}$, $\omega \in \Omega$, then for every fixed $r \in Q$ the function $L(r,\cdot;f,T)$ belongs to $\widetilde{H}^{\omega}$ also.*

Applying Lemma 1 and Lemma 4, we shall prove two theorems.

THEOREM 3. *If $f \in H^{\omega}$, $\omega \in \Omega$, then*

$$\|L(r,\cdot;f,T)\|_C \le M_9 \|f\|_{\omega} \sum_{n=0}^{\infty} 2^n T_{\overline{n},2\overline{n}}(r)\omega(2^{-n})$$

for all $r \in Q$ ($\overline{n} = 2^n - 1$).

Proof. Using Lemma 1 and (6) to (17), we obtain

$$\begin{aligned}\|L(r,\cdot;f,T)\|_C &\le \sum_{n=0}^{\infty} 2^n \|W_{2\bar{n},\bar{n}}(r,\cdot;f,T)\|_C \\ &\le M_2(2)\sum_{n=0}^{\infty} 2^n T_{\bar{n},2\bar{n}}(r) E_{\bar{n}}(f;C) \\ &\le 3M_2(2)\|f\|_\omega \sum_{n=0}^{\infty} 2^n T_{\bar{n},2\bar{n}}(r)\omega(2^{-n}) \quad \text{for } r\in Q.\end{aligned}$$

THEOREM 4. *Suppose that $f \in H^\omega$, $\omega \in \Omega$, and $\mu \in \Omega$ is a function such that $q(h) = \frac{\omega(h)}{\mu(h)}$ is non-decreasing for $h > 0$. Then*

$$\|L(r,\cdot;f,T)\|_{H^\mu} \le M_{10}(\mu)\|f\|_\omega \sum_{n=0}^{\infty} 2^n T_{\bar{n},2\bar{n}}(r) q(2^{-n}) \tag{19}$$

for all $r \in Q$ $(\overline{n} = 2^n - 1)$.

Proof. By Theorem 1 and (5) we have $L(r,\cdot;f,T) \in H^\mu$ for every fixed $r \in Q$. From this and from (17) we get

$$\|L(r,.;f,T)\|_{H^\mu} \le \sum_{n=0}^{\infty} 2^n \|W_{2\bar{n},\bar{n}}(r,\cdot;f,T)\|_{H^\mu}$$

$(r \in Q)$. Now using Lemma 4, we obtain (19).

4. Applications

4.1. **Riesz method**. Consider the following strong Riesz means of Fourier series of $f \in C$:

$$R_n(x;f,\beta) := \frac{1}{(n+1)^\beta}\sum_{k=0}^{n}((k+1)^\beta - k^\beta)|S_k(x;f) - f(x)|,$$

$n \in N$, $x \in R$, $\beta > 0$. From Theorem 1 and Theorem 4 we obtain the following

COROLLARY 1. *Suppose that $f \in H^\omega$, $\omega \in \Omega$, and μ is a function as in Theorem 4. Then, for every fixed $\beta > 0$, we have*

$$\|R_n(\cdot;f,\beta)\|_{H^\mu} \le M_{11}(\mu,\beta)\|f\|_\omega (n+1)^{-\beta}\sum_{k=0}^{m} 2^{k\beta} q(2^{-k})$$

for all $n \in N$, where m is integer such that $2^m \le n+1 < 2^{m+1}$.

In particular, if $q(h) \le M_{12}h^{\gamma}$ $(0<\gamma<1)$ for $h>0$, then

$$\|R_n(\,\cdot\,;f,\beta)\|_{H^{\mu}} \le M_{13}^{*}\|f\|_{\omega} \begin{cases} (n+1)^{-\gamma} & \text{if } \gamma<\beta, \\ (n+1)^{-\gamma}\log(n+1) & \text{if } \gamma=\beta, \\ (n+1)^{-\beta} & \text{if } \gamma>\beta, \end{cases}$$

for all $n \in N$, where $M_{13}^{*} = M_{13}(\mu,\beta,\gamma)$.

Using Theorem 2, (20) and Lemma 6 to (21), we obtain

COROLLARY 2. *Suppose that ω and μ are two functions as in Theorem 4 and $q(h) \le M_{12}h^{\gamma}$ $(0<\gamma<1)$ for $h>0$. If $f \in \widetilde{H}^{\omega}$, then*

$$\|R_n(\,\cdot\,;f,\beta)\|_{\tilde{H}^{\mu}} = \begin{cases} o((n+1)^{-\gamma}) & \text{if } 0<\gamma<\beta, \\ o((n+1)^{-\gamma}\log(n+1)) & \text{if } \gamma=\beta, \\ O((n+1)^{-\beta}) & \text{if } \gamma>\beta, \end{cases}$$

as $n \to \infty$.

4.2. **Abel method.** Consider the strong Abel means of Fourier series of $f \in C$:

$$A_p(r,x;f) := (1-r)^{1+p}\sum_{k=0}^{\infty}\binom{k+p}{k} r^k |S_k(x;f)-f(x)|, \tag{22}$$

$x \in R$, $r \in (0,1)$, $r \to 1_-$ and $p \in N$.

It is easily verified that the sequence T defining the strong Abel means (22) satisfies the conditions 1°–4° and

$$\begin{aligned} T_{\bar{n},2\bar{n}}(r) &= \max_{\bar{n}\le k\le 2\bar{n}} (1-r)^{1+p}\binom{k+p}{k} r^k \\ &\le \frac{(p+1)^p}{p!}(1-r)^{1+p}(2\overline{n})^p r^{\bar{n}} \end{aligned} \tag{23}$$

for $n \in N$, $\overline{n} = 2^n - 1$, $p \in N$ and $r \in (0,1)$.

Using Theorem 1, Theorem 4 and (23) to (22), we obtain

COROLLARY 3. *Under the assumptions of Theorem 4 we have*

$$\|A_p(r,\,\cdot\,;f)\|_{H^{\mu}} \le M_{14}^{*}\|f\|_{\omega}(1-r)^{1+p}\sum_{k=0}^{S}2^k(p+1)q(2^{-k}))$$

for all $r \in (0,1)$, where $S = [\log_2 \frac{1}{1-r}]$ (i.e. S is the integral part of $\log_2 \frac{1}{1-r}$) and $M_{14}^{*} = M_{14}(\mu,p)$.

In particular if $g(h) \le M_{12}h^{\gamma}$ for $h>0$ with $0<\gamma<1$, then

$$\|A_p(r,\,\cdot\,;f)\|_{H^{\mu}} \le M_{15}(\mu,p,\gamma)\|f\|_{\omega}(1-r)^{\gamma}$$

for all $r \in (0,1)$.

From Theorem 2, (5), (20)–(22) and Lemma 6 follows

COROLLARY 4. *Let ω and μ be a functions as in Corollary 2. If $f \in H^\omega$ and $p \in N$, then*

$$\|A_p(r, \cdot; f)\|_{\tilde{H}^\mu} = o((1-r)^\gamma) \quad as\ r \to 1_-.$$

4.3. **Riemann method.** Now we shall consider the strong Riemann means of Fourier series (8) of $f \in C$ defined by the formula

$$R(r, x; f) = \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{\sin^2 kr}{k^2 r} |S_k(x; f) - f(x)|,$$

$x \in R$, $0 < r < 1$, $r \to 0_+$ ([7], [8]).

The sequence T with $t_k(r) = \frac{\sin^2 kr}{k^2 r}$, $0 < r < 1$, $r \to 0_+$, satisfies the conditions 1°–4°, e.g. applying the inequality

$$\frac{\sin^2 kr}{k^2 r} \le \begin{cases} r & \text{if } 0 < kr \le 1, \\ \frac{1}{k^2 r} & \text{if } kr > 1, \end{cases}$$

and writing $S = [\log_2(\frac{1}{r} + 2)]$, $0 < r < 1$, we get

$$\sum_{n=1}^{\infty} 2^n T_{\bar{n}, 2\bar{n}}(r) \le \Big(\sum_{n=1}^{S-1} + \sum_{n=S}^{\infty} \Big) 2^n T_{\bar{n}, 2\bar{n}}(r)$$

$$\le r \sum_{n=1}^{S-1} 2^n + \frac{1}{r} \sum_{n=S}^{\infty} 2^n (2^n - 1)^{-2} \le 35,$$

which proves that the condition 4° is fulfilled.

From Theorems 1–4 and Lemma 1–6 we obtain the following estimations:

COROLLARY 5. *If $f \in H^\omega$, $\omega \in \Omega$, then*

$$\|R(r, \cdot; f)\|_C \le M_{16} \|f\|_\omega r \sum_{n=1}^{S} 2^n \omega(2^{-n})$$

for all $r \in (0, 1)$, where $S = [\log_2(\frac{1}{r} + 2)]$.

COROLLARY 6. *If the assumptions of Theorem 4 are satisfied, then*

$$\|R(r, \cdot; f)\|_{H^\mu} \le M_{17}(\mu) \|f\|_\omega r \sum_{n=1}^{S} 2^n q(2^{-n})$$

for all $r \in (0, 1)$, where $S = [\log_2(\frac{1}{r} + 2)]$.

In particular, if $g(h) \le M_{12} h^\gamma$, $0 < \gamma < 1$, for $h > 0$, then

$$\|R(r, \cdot; f)\| H^\mu \le M_{18}(\mu, \gamma) \|f\|_\omega r^\gamma$$

for all $r \in (0, 1)$.

COROLLARY 10. *If the assumptions of Corollary 2 are satisfied, then for* $f \in \widetilde{H}^{\mu}$ *we have*

$$\|R(r, \cdot\,; f)\|_{\widetilde{H}^{\mu}} = o(r^{\gamma}) \quad \text{as } r \to 0_{+}.$$

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ON-LINE COVERING THE UNIT SQUARE BY SQUARES AND THE THREE-DIMENSIONAL UNIT CUBE BY CUBES

We say that a sequence $C_1, C_2, \ldots$ of sets of Euclidean d-space E^d *permits a covering* of a set $C \subset E^d$ if there are rigid motions σ_i such that C is contained in the union of sets $\sigma_i C_i$ for $i = 1, 2, \ldots$ A survey of results on the covering of convex bodies by sequences of convex bodies can be found in the paper [2] of Groemer. We consider *on-line* covering methods in which we are given C_i, where $i > 1$, only after the motion σ_{i-1} has been provided; at the beginning we are given C_1. The on-line restriction is introduced here by analogy to the on-line restriction for packing problems (see [1] and [5]).

The first solution of an on-line covering problem was given by Kuperberg [4] who proved that the unit cube I^d of E^d can be on-line covered by every sequence of cubes of the total volume at least 4^d. The paper [3] presents two on-line methods of a covering of I^d by cubes. The first method permits an on-line covering of I^d by every sequence of cubes of the total volume at least $3 \cdot 2^d - 4$. The second method is much more efficient. For $d > 3$ it enables an on-line covering of the unit cube by arbitary sequence of cubes whose sum of volumes is at least $2^d + (\frac{4}{3}\sqrt{2} + \frac{2}{3})(\frac{3}{2})^d$. In the present paper we consider the cases $d = 2$ and $d = 3$. We apply an improvement of the second method. The unit square can be covered by this improved method if the total area of a sequence of squares used for the covering is at least $\frac{7}{4} \cdot \sqrt[3]{9} + \frac{13}{8} \cong 5.265$. The three-dimensional cube can be covered by this method by every sequence of cubes of the total volume at least $\frac{143}{32} + \frac{3}{8}(\frac{9}{2})^{3/4} + \frac{63}{8}(\frac{2}{9})^{1/4} \cong 11.034$.

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1. Description of the method

Let p and q be real numbers such that $pq = 2$ and that $\frac{4}{3} \le p < 2$. Clearly, $1 < q \le \frac{3}{2}$.

We will on-line cover the unit cube

$$I^d = \{(x_1, \dots, x_d);\ 0 \le x_1 \le 1, \dots, 0 \le x_d \le 1\}$$

by a sequence $Q_1, Q_2, \dots$ of cubes. We call a cube of our sequence *very big* if its sides are of length at least 1, and we call it *big* if its sides are at least $\frac{1}{2}p$ but smaller than 1.

There is a cube N_i in Q_i whose side is of the greatest possible length of the form 2^{-r} or $p \cdot 2^{-r-2}$, where r is a non-negative integer. If Q_i is not big and if the length of the side of N_i is of the form 2^{-r}, where r is a non-negative integer, then we put $B_i = N_i$ and $T_i = \emptyset$. If Q_i is big, then let us denote by h_i the length of the side of Q_i and let us find in Q_i a rectangular parallelotope M_i containing N_i and having $d-1$ perpendicular sides of length $\frac{1}{2}$ and the side of length h_i perpendicular to them; we put $B_i = M_i$ and $T_i = \emptyset$. If the side of N_i is of the form $p \cdot 2^{-r-2}$, where r is a non-negative integer, we put $B_i = \emptyset$ and $T_i = N_i$.

For every non-empty B_i we find the greatest number $b_i \le 1$ such that every point of I^d whose last coordinate is smaller than b_i has been covered by cubes Q_j such that $j < i$. The set of all points of the cube I^d having the last coordinate b_i is called *the i-th bottom* of the cube. Of course, if $b_i > 0$, then the i-th bottom is covered by cubes Q_j, where $j < i$. A point of the *i-th* bottom is called *surface* if there is a point in I^d with the last coordinate greater and with unchanged the other coordinates which has not been covered by cubes Q_j, where $j < i$. We find a rigid motion σ_i such that $\sigma_i B_i$ contains a surface point of the i-th bottom and that it has the form

$$\{(x_1, \dots, x_d);\ a_k 2^{-r} \le x_k \le (a_k + 1)2^{-r} \text{ for } k = 1, \dots, d-1 \text{ and } b_i \le x_d \le b_i + 2^{-r}\},$$

where 2^{-r} denotes the length of the side of the cube N_i, and where the numbers $a_1, \dots, a_{d-1}$ belong to the set $\{0, \dots, 2^r - 1\}$. Let us add that if Q_i is big, we find a rigid motion σ_i such that $\sigma_i Q_i$ is between the hyperplanes $x_k = 0$ and $x_k = 1$ for $k = 1, \dots, d-1$, and that $\sigma_i B_i$ contains a surface point of the i-th bottom and that it has the form

$$\{(x_1, \dots, x_d);\ a_k 2^{-1} \le x_k \le (a_k + 1)2^{-1} \text{ for } k = 1, \dots, d-1 \text{ and } b_i \le x_d \le b_i + h_i\},$$

where $a_1, \dots, a_{d-1} \in \{0, 1\}$. As long as I^d is not covered, every i-th bottom contains surface points. We say that a big cube Q_i is *early* if $\sigma_i B_i \subset I^d$ and if $b_i + h \le b_m$ for an $m > i$, and we say that it is *late* in the opposite case.

The sequence $T_1, T_2, \ldots$ is used for covering I^d by a similar method. For every T_i we find the greatest number $t_i \leq 1$ such that all points of I^d with the last co-ordinate greater than $1 - t_i$ are covered by cubes Q_j, where $j < i$. The set of all points of I^d with the last coordinate equal to $1 - t_i$ is called *the i-th top*. A point of the i-th top is called *surface* if there is a point in I^d with the last coordinate smaller and with unchanged the other coordinates which has not been covered by cubes Q_j, where $j < i$. If $T_i \neq \emptyset$, we find a rigid motion σ_i such that $\sigma_i T_i$ contains a surface point of the i-th top and that it is of the form

$$\{(x_1, \ldots, x_d);\ a_k p 2^{-r-2} \leq x_k \leq (a_k + 1) p 2^{-r-2} \text{ for } k = 1, \ldots, d-1 \\ \text{and } 1 - t_i - p2^{-r-2} \leq x_d \leq 1 - t_i\},$$

where $p \cdot 2^{-r-2}$ denotes the length of the side of the cube T_i and where $a_1, \ldots, a_k \in \{0, 1, \ldots, 3 \cdot 2^r - 1\}$. As long as I^d is not covered, every i-th top contains surface points.

So for every index i a rigid motion σ_i is found such that the non-empty set from the sets B_i or T_i is moved in order to cover our unit cube. Together with the set we move the corresponding cube Q_i. If $b_n + t_n \geq 1$, then we can stop the covering process because I^d has been covered by cubes $\sigma_1 Q_1, \ldots, \sigma_{n-1} Q_{n-1}$.

Remark. Our method in the case when $p = q = \sqrt{2}$ is an improvement of the method of the current bottom and top considered in [3]. We change here the definitions of the i-th bottom, of the i-th top, and of the surface points. Moreover, we precisely determine where to put big cubes. The introduced improvements eliminate putting a cube in a place totally covered by previous cubes.

From the comparison of the presented method with the method of the current bottom and top we conclude that Lemmas 1 and 2 of [3] can be generalized here as follows

LEMMA 1. *The set*

$$\{(x_1, \ldots, x_d);\ 0 \leq x_1 \leq 1, \ldots, 0 \leq x_d \leq a\}$$

can be covered by our method by every sequence of cubes whose sides are of the form 2^{-r}, *where* $r \in \{1, 2, \ldots\}$, *if the total volume of the cubes is at least*

$$b(a) = \frac{2^d - 1}{2^{d-1}} \left[a + \frac{2^{d-1} - 1}{2^d - 1} \right].$$

LEMMA 2. *The set*

$$\{(x_1, \ldots, x_d);\ 0 \leq x_1 \leq 1, \ldots, a \leq x_d \leq 1\}$$

can be covered by our method by every sequence of cubes whose sides are of the form $p \cdot 2^{-r-1}$, where $r \in \{1, 2, \ldots\}$, if the total volume of the cubes is at least

$$t(a) = \frac{2^d - 1}{2^{d-1}} \left[(1-a) \left(\frac{3p}{4} \right)^{d-1} + \left(3^{d-1} - \frac{2^{d-1}}{2^d - 1} \right) \left(\frac{p}{4} \right)^d \right].$$

2. Covering the unit square by a sequence of squares

THEOREM 1. *The unit square can be on-line covered by every sequence of squares of the total area at least*

$$\frac{7}{4}\sqrt[3]{9} + \frac{13}{8} \cong 5.265. \tag{1}$$

Proof. Applying the method described in the preceding part, where $p = \sqrt[3]{3}$, we on-line cover the unit square I^2 by arbitrary sequence $Q_1, Q_2, \ldots$ of squares. If in the sequence there is a very big square or if there are at least four big squares, then the sequence covers the whole I^2. So let us assume further that there is no very big square and that the sequence contains at most three big squares.

Let us explain that the value $\sqrt[3]{3}$ for p has been chosen in order to minimize the maximum of the numbers $p^2 b(0) + q^2 t(0)$ and $p^2 t(1) + q^2 t(1)$. It is easy to check that for $p = \sqrt[3]{3}$ the value of $p^2 b(a) + q^2 t(a)$ does not depend on a and that

$$p^2 b(a) + q^2 t(a) = 2\sqrt[3]{9} + \frac{7}{8}. \tag{2}$$

Observe that every big square Q_i contains a square Q_i' of sides of length $\frac{1}{2}p$ such that

$$|Q_i| - |Q_i'| < 1 - \frac{1}{4}p^2 \quad \text{and} \quad |Q_i'| = p^2 |N_i|. \tag{3}$$

Case 1, *in which the sequence $Q_1, Q_2, \ldots$ does not contain big squares.*

From Lemmas 1 and 2, and from (2) we see that I^2 can be covered by every sequence $Q_1, Q_2, \ldots$ of the total area at least $2\sqrt[3]{9} + \frac{7}{8} \simeq 5.035$.

Case 2, *in which the sequence $Q_1, Q_2, \ldots$ contains exactly one big square.*

Denote by Q_m the unique big square in our sequence.

For $d = 2$ the value of $t(a)$ defined in Lemma 2 is

$$\frac{3}{2}\left[(1-a) \cdot \frac{3}{4}p + \frac{7}{48}p^2 \right]. \tag{4}$$

Particularly, the value $\frac{7}{48}p^2$ in (4) originates from

$$\left(\frac{p}{4}\right)^2 + \left(\frac{p}{4}\right)^2 + \left(\frac{p}{8}\right)^2 + \left(\frac{p}{16}\right)^2 + \ldots = \frac{7}{48} \cdot p^2 \tag{5}$$

It estimates from above the area "under" the i-th top which is covered by squares $\sigma_j T_j$ for $j < i$.

Below we consider two subcases; when the difference $t_m - b_m$ is smaller than $\frac{1}{4}p$, and when it is at least $\frac{1}{4}p$.

Assume that $t_m - b_m < \frac{1}{4}p$. Clearly, at most two of the three rectangles

$$\left\{(x,y);\ \frac{1}{4}pk \le x \le \frac{1}{4}p(k+1),\ b_m \le y \le t_m\right\}, \qquad \text{where } k = 0, 1, 2,$$

has been covered by squares $\sigma_j Q_j$ for $j < m$. If two of the above three rectangles has been covered by squares $\sigma_j Q_j$ for $j < m$, then the third of them is covered by $\sigma_m Q_m$. So since now we assume that at most one of the three rectangles has been covered by squares $\sigma_j Q_j$ for $j < m$. Observe that for $d = 2$ the total area of squares $\sigma_1 B_1, \sigma_2 B_2, \ldots$ in Lemma 1 is smaller than $\frac{3}{2}(a + \frac{1}{3})$, and the total area of squares $\sigma_1 T_1, \sigma_2 T_2, \ldots$ in Lemma 2 is smaller than $\frac{3}{2}[(1-a) \cdot \frac{3}{4}p + \frac{7}{48}p^2]$. So from the inequality $t_m - b_m < \frac{1}{4}p$ we conclude that the total area of the squares $\sigma_1 N_1, \sigma_2 N_2, \ldots$ is less than $\frac{3}{2}[a + \frac{1}{3} + (1-a)\frac{3}{4}p + \frac{7}{48}p^2 - \frac{1}{2}(t_m - b_m)]$. Moreover, instead of one of the components $(\frac{1}{4}p)^2$ in (5) we have now another component; $\frac{1}{4}p(t_m - b_m)$ or $\frac{1}{3}(\frac{1}{4}p)^2$. This and (3) imply that the difference between the loss and the profit (in comparison to the situation when we would not have this square) in the total area of the sequence of squares which permits a covering of the square I^2 is not smaller than $1 - (\frac{1}{2}p)^2 - \frac{3}{2}[\frac{1}{2}(t_m - b_m) + (\frac{1}{4}p)^2 - \frac{1}{4}p(t_m - b_m)] \cdot q^2$ for the component $\frac{1}{4}p(t_m - b_m)$, or at least $1 - (\frac{1}{2}p)^2 - \frac{3}{2}[\frac{1}{2}(t_m - b_m) + (\frac{1}{4}p) - \frac{1}{3}(\frac{1}{4}p)^2] \cdot q^2$ for the component $\frac{1}{3}(\frac{1}{4}p)^2$. Observe that both the numbers are not greater than $1 - \frac{1}{4}\sqrt[3]{9} - \frac{1}{4} = \frac{3}{4} - \frac{1}{4}\sqrt[3]{9}$.

Now assume that $t_m - b_m \ge \frac{1}{4}p$. Similarly like in the preceding paragraph we get that the total area of squares $\sigma_1 N_1, \sigma_2 N_2, \ldots$ is smaller than $\frac{3}{2}[a + \frac{1}{3} + (1-a)\frac{3}{4}p + \frac{7}{48}p^2 - \frac{1}{2}(t_m - b_m)]$ if $t_m - b_m \le \frac{1}{2}p$, and that it is smaller than $\frac{3}{2}[a + \frac{1}{3} + (1-a)\frac{3}{4}p + \frac{7}{48}p^2 - \frac{1}{2}(\frac{1}{2}p - \frac{1}{2})]$ if $t_m - b_m > \frac{1}{2}p$. Similarly we observe that in both situations the diference between the loss and the profit (in comparison to the situation when we would not have this square) in the total area of the sequence of cubes which can cover I^2 is smaller than $\frac{3}{4} - \frac{1}{3}\sqrt[3]{9}$.

Since in each of the above subcases the difference between the loss and the profit is not greater than $\frac{3}{4} - \frac{1}{4}\sqrt[3]{9}$, then I^2 can be covered by our method provided the total area of the squares in the sequence is at least $\frac{3}{4} - \frac{1}{4}\sqrt[3]{9} + 2\sqrt[3]{9} + \frac{7}{8} = \frac{13}{8} + \frac{7}{4}\sqrt[3]{9} \cong 5.265$.

Case 3, *in which there are exactly two big squares in the sequence* $Q_1, Q_2, \ldots$

At least one of the two big squares must be an early square. Denote it by Q_m. Put $S_m = M_m \setminus N_m$. We have $|S_m| = \frac{1}{2}(h_m - \frac{1}{2})$. Since the rectangle $\sigma_m S_m$ is a subset of I^2 such that $b_m + h_m \le b_w$ for an index $w > m$, and since it covers only points not covered earlier in the covering process, it replaces a number of squares of the form Q_j counted in (2) whose sum of volumes is $\frac{1}{2}(h_m - \frac{1}{2}) \cdot \frac{3}{2}p^2$. So when we have an early big square in the sequence, the diference between the loss and the profit (in comparison to the situation when we would not have this square) in the total area of the sequence of cubes which is able to cover I^2 is

$$h_m^2 - \left(\frac{1}{2}p\right)^2 - \frac{1}{2}\left(h_m - \frac{1}{2}\right) \cdot \frac{3}{2}p^2,$$

which is not greater than the negative number $1 - \frac{5}{8}\sqrt[3]{9}$.

We see that I^2 permits a covering by our method if the total area of the squares of the sequence $Q_1, Q_2, \ldots$ is at least $1 - (\frac{1}{2}p)^2 + 1 - \frac{5}{8}\sqrt[3]{9} + 2\sqrt[3]{9} + \frac{7}{8} \cong 5.215$.

Case 4, *in which there are exactly three big squares in the sequence* $Q_1, Q_2, \ldots$

Since two of the three squares are early, I^2 permits a covering by our method if the total area of the squares of the sequence $Q_1, Q_2, \ldots$ is at least $1 - (\frac{1}{2}p)^2 + 2(1 - \frac{5}{8}\sqrt[3]{9}) + 2\sqrt[3]{9} + \frac{7}{8} \simeq 4.915$. ■

3. Covering the three-dimensional unit cube by a sequence of cubes

THEOREM 2. *The cube I^3 permits an on-line covering by every sequence of three-dimensional cubes of the total volume at least*

$$\frac{143}{32} + \frac{3}{8}\left(\frac{9}{2}\right)^{3/4} + \frac{63}{8}\left(\frac{2}{9}\right)^{1/4} \cong 11.034.$$

Proof. Applying our method for $p = (\frac{9}{2})^{1/4}$ we on-line cover I^3 by arbitrary sequence $Q_1, Q_2, \ldots$ of cubes. Let us assume that there is no very big cube and that the sequence contains at most seven big cubes; in the opposite case the thesis is trivially true. It is easy to check that for $p = (\frac{9}{2})^{1/4}$ the value of $p^3 b(a) + q^3 t(a)$ does not depend on a and that $p^3 b(a) + q^3 t(a) = \frac{3}{4}p^3 + \frac{63}{8}p^{-1} + \frac{59}{32}$.

Case 1, *in which the sequence $Q_1, Q_2, \ldots$ does not contain big cubes.* Of course, the cube I^3 permits a covering by our method if the total

volume of cubes of the sequence $Q_1, Q_2, \ldots$ is at least $\frac{3}{4}p^3 + \frac{63}{8}p^{-1} + \frac{59}{32} \cong$ 9.568.

Case 2, *in which the sequence $Q_1, Q_2, \ldots$ contains at least one big cube.* Observe that for every early big cube Q_m the difference between the lose and the profit (in comparison to the situation when we would not have this cube) is equal to

$$h_m^3 - \left(\frac{1}{2}p\right)^3 - \left(\frac{1}{2}\right)^2 \left(h_m - \frac{1}{2}\right) p^2 \cdot \frac{7}{4}.$$

This number is not greater than $1 - \frac{11}{32}p^3$, and thus it is negative.

Of course, our sequence contains at most three late big cubes. Analogically like in Case 2 of Theorem 2 we can show that for every late big cube the difference between the lose and the profit is equal to $1 - \frac{1}{8}p^3 - \frac{1}{8}$.

We conclude that I^3 can be covered by our method by every sequence of cubes of the total volume at least

$$3\left(1 - \frac{1}{8}p^3 - \frac{1}{8}\right) + \frac{3}{4}p^3 + \frac{63}{8}p^{-1} + \frac{59}{32} = \frac{143}{32} + \frac{3}{8}\left(\frac{9}{2}\right)^{\frac{3}{4}} + \frac{63}{8}\left(\frac{2}{9}\right)^{\frac{1}{4}} \cong 11.034. \blacksquare$$

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A MEASURE OF NONCOMPACTNESS IN SEQUENCE BANACH SPACES

In this paper we define a measure of noncompactness (m.n.c.), in some sequence Banach spaces, which is equivalent to the Hausdorff m.n.c.. In [3] another m.n.c. in those spaces is considered, but it is not equivalent to the Hausdorff m.n.c.. The notion of a m.n.c. turns out to be an useful tool in many branches of mathematical analysis. The current state of this theory and its applications can be found in the books [1] and [2].

Assume that X is a Banach space. The unit closed ball will be denoted by B_X. Moreover $Conv(A)$ denotes the convex closure of a set $A \subset X$ and $\|A\| = sup\{\|x\| : x \in A\}$. Finally, denote by $P_b(X)$ the family of all nonempty and bounded subsets of X. We use the following definition [2]: a function

$$\mu : P_b(X) \to [0, \infty[$$

will be called a m.n.c. in X if it satisfies the following conditions:

(1) $\mu(A) = 0 \Leftrightarrow A$ is relatively compact,
(2) $A \subset B \Rightarrow \mu(A) \leq \mu(B)$,
(3) $\mu(\mathrm{Conv}\,(A)) = \mu(A)$,
(4) $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$,
(5) $\mu(A + B) \leq \mu(A) + \mu(B)$,
(6) $\mu(cA) = |c|\mu(A)$ (for every scalar c).

A m.n.c. that it seems to be the most convenient in the applications is the Hausdorff m.n.c. $h : P_b(X) \to [0, \infty[$, defined in the following way, for $A \subset X$ nonempty and bounded

$$h(A) := \inf\{r > 0 : \exists F \subset X \text{ finite}, \ A \subset F + rB_X\}.$$

This m.n.c. verifies $h(B_X) = 1$.

Assume that (X_i) is a sequence of Banach spaces. Denote by $l^p(X_i)$, $1 \leq p < \infty$, the space of all sequences $x = (x_i)$, $x_i \in X_i$ for $i = 1, 2, \ldots$ such

that

$$\sum_{i=1}^{\infty} \|x_i\|^p < \infty.$$

It is well known that $l^p(X_i)$ is a Banach space under the norm

$$\|(x_i)\| = \Big(\sum_{i=1}^{\infty} \|x_i\|^p\Big)^{1/p}.$$

Denote by h^p the Hausdorff m.n.c. on the space $l^p(X_i)$ and h_i the Hausdorff m.n.c. on the space X_i. We consider the following operators

$$\pi_n : (x_i) \in l^p(X_i) \to x_n \in X_n,$$
$$\tau_n : (x_i) \in l^p(X_i) \to (0, \ldots, 0, x_n, x_{n+1}, \ldots) \in l^p(X_i),$$

and the following quantities, for $A \in P_b(l^p(X_i))$:

$$a^p(A) := \sup\{h_n(\pi_n(A)) : n = 1, 2, \ldots\},$$
$$b^p(A) := \inf\{\|\tau_n(A) : n = 1, 2, \ldots\}.$$

In [3], based on a theorem of [4], it is proved that

$$\mu^p(A) := \max\{a^p(A), b^p(A)\}$$

defines a m.n.c. on the space $l^p(X_i)$, which is not equivalent to the Hausdorff m.n.c. h^p. Now we consider the operator

$$\sigma_n : (x_i) \in l^p(X_i) \to (x_1, \ldots, x_n, 0, 0, \ldots) \in l^p(X_i)$$

and the quantity

$$c^p(A) := \sup\{h(\sigma_n(A)) : n = 1, 2, \ldots\}.$$

Note that $h(\sigma_n(A))$ agrees with the Hausdorff m.n.c. of $\sigma_n(A)$ in the space $l^p(X_1, X_2, \ldots, X_n)$. In the following theorem we define a m.n.c. η^p in the space $l^p(X_i)$, which is equivalent to Hausdorff m.n.c. h^p.

THEOREM. *In the space $l^p(X_i)$ we define the quantity*

$$\eta^p(A) := \max\{c^p(A), b^p(A)\},$$

for A nonempty and bounded subset of $l^p(X_i)$. Then η^p is a measure of noncompactness and moreover

$$\eta^p \le h^p \le 2\eta^p(A).$$

Proof. The proof of the first part is very simple and is omitted. To prove the second part denote $r := h^p(A)$. Then, for $\varepsilon > 0$, we can find a finite set $F \subset l^p(X_i)$ such that

$$A \subset F + (r+\varepsilon)B_{l^p(X_i)}.$$

Using the equality

$$\eta^p(B_{l^p(X_i)}) = 1$$

and the properties of a m.n.c., we infer that $\eta^p(A) \le r+\varepsilon$. The arbitrariness of ε completes a part of the proof: $\eta^p \le h^p$.

In what follows we show that $h^p \le 2\eta^p$. Obviously, we have, for any $A \subset l^p(X_i)$ nonempty and bounded,

$$A \subset \sigma_n(A) + \tau_{n+1}(A) \qquad (n = 1, 2, \ldots).$$

We put $r := \eta^p(A)$. Consequently, we have $b^p(A) \le r$ and $c^p(A) \le r$. Given any $\varepsilon > 0$, by definition of $b^p(A)$, choose m such that $\|\tau_m(A)\| < r+\varepsilon$. On the other hand, by definition of $c^p(A)$, we have $h^p(\sigma_n(A)) \le r$, for $n = 1, 2, \ldots$ Since

$$A \subset \sigma_m(A) + \tau_{m+1}(A),$$

and by the properties of a m.n.c., we obtain that

$$h^p(A) \le h^p(\sigma_m(A)) + h^p(\tau_{m+1}(A)).$$

Trivially $h^p(A) \le \|A\|$, hence

$$h^p(A) \le r + \|\tau_{m+1}(A)\| < 2r + \varepsilon.$$

The arbitrariness of ε gives

$$h^p(A) \le 2r = 2\eta^p(A)$$

and it completes the proof.

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A THEOREM OF THE HAHN-BANACH TYPE

Let Y be a linear subspace of a linear space X over the rationals $\mathbb{Q}$ and let $C \subseteq X$ be $\mathbb{Q}$-convex. Moreover, let $\mathcal{F}$ be a family of subsets of a linear space E over $\mathbb{Q}$ having the binary intersection property. Suppose that F is a $\mathbb{Q}$-concave set-valued function defined on C and assuming values in $\mathcal{F}$. We give some conditions under which every additive selection of the restriction of F to $Y \cap C$ can be extended to an additive selection of F.

1. Let X be a linear space over the set of rational numbers $\mathbb{Q}$ and let $A \subseteq X$ be a set. We say that A is *$\mathbb{Q}$-radial at a point* $a \in A$ iff for every $x \in X, x \neq 0$ there exists an $\varepsilon > 0$ such that $a + \lambda x \in A$ for every $\lambda \in (-\varepsilon, \varepsilon) \cap \mathbb{Q}$.

A non-empty set $A \subseteq X$ is called *$\mathbb{Q}$-convex* iff $\lambda x + (1-\lambda)y \in A$ for all $x, y \in A$ and $\lambda \in \mathbb{Q} \cap [0,1]$. A functional $p : A \to \mathbb{R}$ defined on a $\mathbb{Q}$-convex set A is called $J-$ *convex* iff

(1)
$$p\left(\frac{x+y}{2}\right) \leq \frac{p(x)+p(y)}{2} \quad \text{for } x, y \in A.$$

The proof of the following theorem can be found in [2] (Theorem 10.1.1) where X is $n-$dimensional euclidean space $\mathbb{R}^n$. The proof in the general case differs from that one only formally.

THEOREM A. *Let $D \subseteq X$ be a $\mathbb{Q}$-convex and $\mathbb{Q}$-radial at a point $x_0 \in D$. Assume that $Y \subseteq X$ is a linear subspace over $\mathbb{Q}$ of X, $x_0 \in Y$ and $p : D \to \mathbb{R}$ is a J-convex function. If $f : Y \to \mathbb{R}$ is an additive function fulfilling*

(2)
$$f(x) \leq p(x) \quad \text{for } x \in D \cap Y,$$

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then there exists an additive function $g : X \to \mathbb{R}$ such that $g|_Y = f$ and

$$g(x) \leq p(x) \quad \textit{for } x \in D.$$

Suppose that the hypotheses of Theorem A hold. Put

$$C := (D - x_0) \cap (x_0 - D)$$

and

$$q(x) := p(x_0 + x) - f(x_0), \quad \text{for } x \in C.$$

We can observe that C is symmetric, $\mathbb{Q}$−convex, and $\mathbb{Q}$−radial at 0 and q is J-convex on C. Moreover the inequalities

$$(2') \qquad f(x) \leq q(x) \quad \text{for } x \in C \cap Y,$$

$$(3') \qquad g(x) \leq q(x) \quad \text{for } x \in C,$$

hold. Setting in $(2')$ $x = 0$, we get $0 = f(0) \leq q(0)$. Next putting in (1) for the functional q, $y = -x$, $x \in C$ we obtain

$$0 \leq q(0) \leq \frac{1}{2}q(x) + \frac{1}{2}q(-x).$$

Consequently

$$-q(-x) \leq q(x) \quad \text{for all } x \in C.$$

Now we can introduce a set-valued function on C with compact and convex values in $\mathbb{R}$ by the formula

$$F(x) = [-q(-x), q(x)], \quad x \in C.$$

It is easy to check that the set-valued function F fulfils the following conditions:

$$(1'') \qquad F(\lambda x + (1 - \lambda)y) \subseteq \lambda F(x) + (1 - \lambda)F(y)$$

for $x, y \in C$ and for $\lambda \in \mathbb{Q} \cap [0, 1]$,

$$(2'') \qquad f(x) \in F(x) \quad \text{for } x \in C \cap Y,$$

$$(3'') \qquad g(x) \in F(x) \quad \text{for } x \in C.$$

In addition F is an odd set-valued function, i.e.,

$$(3) \qquad F(-x) = -F(x) \quad \text{for } x \in C.$$

Conversely, if the set-valued function F fulfils conditions $(1'')$, $(2'')$ and $(3'')$, then for q the relations (1), $(2')$ and $(3')$ hold.

In the next part of the paper we consider the family $\mathcal{F}$ of subsets of a linear space over $\mathbb{Q}$. We assume that $\mathcal{F}$ has the binary intersection property. It means that every subfamily of $\mathcal{F}$, any two members of which intersect has non-empty intersection (see [3]).

2. In [1] a version of the Hahn-Banach theorem for subadditive set-valued function is proved. In this paper we are going to give a new version of the Hahn-Banach theorem for concave set-valued function i.e., fulfilling the inclusion $(1'')$.

THEOREM 1. *Let X be a linear space over $\mathbb{Q}$, let $C \subseteq X$ be $\mathbb{Q}$-convex, $\mathbb{Q}$-radial at a point $x_0 \in C$ and $C = -C$. Assume that Y is a linear subspace over $\mathbb{Q}$ of X, $x_0 \in Y$. Furthermore, assume that $\mathcal{F}$ is a family of non-empty subsets of a linear space E over $\mathbb{Q}$ having the binary intersection property and fulfilling the conditions:*

$$A \in \mathcal{F},\ u \in \mathrm{E}\ \Rightarrow A + u \in \mathcal{F} \tag{4}$$

$$A \in \mathcal{F},\ \mu \in \mathbb{Q} \cap (0, \infty)\ \Rightarrow \mu A \in \mathcal{F}. \tag{5}$$

If a set-valued function $F : C \to \mathcal{F}$ fulfils conditions $(1'')$, (3) and $f : Y \to E$ is an additive function, which is a selection of the restriction of F to $C \cap Y$ (i.e., $(2'')$ holds), then there exists an additive extension $g : X \to E$ of f fulfilling $(3'')$.

Proof. Denote by Ω the family of all additive maps $\phi : \operatorname{dom}\varphi \to E$ such that $Y \subseteq \operatorname{dom}\varphi \subseteq X$, where $\operatorname{dom}\varphi$ is a linear subspace of X over $\mathbb{Q}$, $\varphi(x) \in F(x)$ for $x \in \operatorname{dom}\varphi \cap C$ and $\varphi(x) = f(x)$ for $x \in Y$. The family Ω is non-empty because f belongs to it. In this family we introduce the partial order "$\prec$" defined by $\varphi \prec \psi$ iff $\operatorname{dom}\varphi \subseteq \operatorname{dom}\psi$ and $\psi|_{\operatorname{dom}\varphi}$ coincides with φ. The family Ω is inductive. To see that take a non-empty chain $\mathcal{C} \subseteq \Omega$. Set $\varphi_{\mathcal{C}}(x) = \varphi(x)$ if $x \in \operatorname{dom}\varphi$ and $\varphi \in \mathcal{C}$. It is easy to see that $\varphi_{\mathcal{C}} \in \Omega$. This function is the upper bound of $\mathcal{C}$. Applying the Kuratowski-Zorn lemma we can gain a maximal element in Ω. It suffices to show that an arbitrary φ belonging to Ω whose domain is different from whole X cannot be maximal in Ω. Take $z \in X \setminus \operatorname{dom}\varphi$. Let Z be a linear subspace over $\mathbb{Q}$ of X spanned by $\operatorname{dom}\varphi$ and z. Choose $x, y \in C \cap \operatorname{dom}\varphi$, $\lambda, \mu \in (0, \infty) \cap \mathbb{Q}$ such that $x + \mu z, y + \lambda z, x - \mu z, y - \lambda z \in C$ (such x, y, λ, μ exist because C is $\mathbb{Q}$–radial at x_0 and $x_0 \in \operatorname{dom}\varphi$). We have

$$\frac{\lambda}{\lambda+\mu}\varphi(x) + \frac{\mu}{\lambda+\mu}\varphi(-y) = \varphi\Big(\frac{\lambda}{\lambda+\mu}x + \frac{\mu}{\lambda+\mu}(-y)\Big)$$
$$\in F\Big(\frac{\lambda}{\lambda+\mu}x + \frac{\mu}{\lambda+\mu}(-y)\Big) = F\Big(\frac{\lambda}{\lambda+\mu}(x+\mu z) + \frac{\mu}{\lambda+\mu}(-y-\lambda z)\Big)$$
$$\subseteq \frac{\lambda}{\lambda+\mu}F(x+\mu z) + \frac{\mu}{\lambda+\mu}F(-y-\lambda z).$$

Hence and by (3) we get

$$0 \in \frac{\lambda}{\lambda+\mu}[F(x+\mu z) - \varphi(x)] - \frac{\mu}{\lambda+\mu}[F(y+\lambda z) - \varphi(y)].$$

Thus

$$0 \in \frac{F(x+\mu z)-\varphi(x)}{\mu} - \frac{F(y+\lambda z)-\varphi(y)}{\lambda}$$

whence

$$\frac{F(x+\mu z)-\varphi(x)}{\mu} \cap \frac{F(y+\lambda z)-\varphi(y)}{\lambda} \neq \emptyset. \tag{6}$$

Similarly the relations

$$\frac{\lambda}{\lambda+\mu}\varphi(x)+\frac{\mu}{\lambda+\mu}\varphi(-y)=\varphi\Big(\frac{\lambda}{\lambda+\mu}x+\frac{\mu}{\lambda+\mu}(-y)\Big)$$

$$\in F\Big(\frac{\lambda}{\lambda+\mu}x+\frac{\mu}{\lambda+\mu}(-y)\Big)=F\Big(\frac{\lambda}{\lambda+\mu}(x-\mu z)+\frac{\mu}{\lambda+\mu}(-y+\lambda z)\Big)$$

$$\subseteq \frac{\lambda}{\lambda+\mu}F(x-\mu z)-\frac{\mu}{\lambda+\mu}F(y-\lambda z)$$

give

$$\frac{F(x-\mu z)-\varphi(x)}{-\mu} \cap \frac{F(y-\lambda z)-\varphi(y)}{-\lambda} \neq \emptyset. \tag{7}$$

We have also

$$\frac{\lambda}{\lambda+\mu}\varphi(-x)+\frac{\mu}{\lambda+\mu}\varphi(-y)=\varphi\Big(\frac{\lambda}{\lambda+\mu}(-x)+\frac{\mu}{\lambda+\mu}(-y)\Big)$$

$$\in F\Big(\frac{\lambda}{\lambda+\mu}(-x)+\frac{\mu}{\lambda+\mu}(-y)\Big)=F\Big(\frac{\lambda}{\lambda+\mu}(-x+\mu z)+\frac{\mu}{\lambda+\mu}(-y-\lambda z)\Big)$$

$$\subseteq \frac{\lambda}{\lambda+\mu}F(-x+\mu z)-\frac{\mu}{\lambda+\mu}F(y+\lambda z).$$

The same argument as above allows to get

$$\frac{F(x-\mu z)-\varphi(x)}{-\mu} \cap \frac{F(y+\lambda z)-\varphi(y)}{\lambda} \neq \emptyset. \tag{8}$$

Conditions (6), (7), (8) and the binary intersection property imply that there exists a $u \in E$ such that

$$u \in \bigcap\Big\{\frac{F(x+\mu z)-\varphi(x)}{\mu} : x \in \operatorname{dom}\varphi \cap C, \mu \in \mathbb{Q}\setminus\{0\},\ x+\mu z \in C\Big\}.$$

Consequently

$$\varphi(x)+\lambda u \in F(x+\lambda z) \quad \text{for } x \in \operatorname{dom}\varphi, \lambda \in \mathbb{Q} \ \text{ such that } \ x+\lambda z \in C.$$

The function $\varphi_0 : Z \to E$ defined by $\varphi_0(x+\lambda z) := \varphi(x)+\lambda u$ is an additive extension of φ different from φ and fulfils the condition

$$\varphi_0(x) \in F(x) \quad \text{for } x \in Z \cap C.$$

Thus φ cannot be a maximal element in Ω. The proof is complete. ■

Let X be a real linear space and $A \subseteq X$. We say that A is *radial at a point* $a \in A$ iff for every $x \in X$, $x \neq 0$ there exists an $\varepsilon > 0$ such that $a + \lambda x \in A$ for all $\lambda \in (-\varepsilon, \varepsilon)$.

Similar considerations as in the proof of Theorem 1 give the following result.

THEOREM 2. *Let X be a real linear space, C be a convex symmetric subset of X, and Y be a subspace of X. Assume that $\mathcal{F}$ is a family of non-empty subsets of a real linear space E having the binary intersection property and fulfilling condition* (4) *and*

$$A \in \mathcal{F},\ \mu \in (0, \infty) \Rightarrow \mu A \in \mathcal{F}.$$

If C is radial at a point $x_0 \in A, x_0 \in Y$ and a set-valued function $F : C \to \mathcal{F}$ is concave and fulfils condition (3), *$f : Y \to E$ is a linear function which is a selection of the restriction F to $Y \cap C$, then there exists a linear extension $g : X \to E$ of f fulfilling* $(3'')$.

3. Let E denote *an ordered real linear space* i.e. E has a binary reflexive and transitive relation "$\leq$" such that

$$y_1 \leq y_2 \Rightarrow \lambda y_1 \leq \lambda y_2 \ \text{ for all } \ y_1, y_2 \in E, \text{ and real } \lambda \geq 0,$$
$$y_1 \leq y_2 \Rightarrow y_1 + y_3 \leq y_2 + y_3 \ \text{ for all } \ y_1, y_2, y_3 \in E.$$

We say that E has *the least upper bound property* (abbreviated: *l.u.b.p.*) iff every non-empty subset A of E which has an upper bound, has least upper bound.

As a consequence of Theorem 2 we can get the following theorem.

COROLLARY. *Let X be a linear space and let Y be a linear subspace of X. Assume that D is a convex subset of X, D is radial at $x_0 \in D, x_0 \in Y$ and $D = 2x_0 - D$. Moreover, assume that E is an ordered real linear space with l.u.b.p. If $p : D \to E$ is convex and $f : Y \to E$ is a linear function dominated by p on $Y \cap D$, then there is a linear extension $g : X \to E$ of f which is dominated by p on D.*

Proof. Consider the family $\mathcal{F}$ of all intervals $[a, b]$ in E, the set $C = (D - x_0) \cap (x_0 - D)$ and the set-valued function

$$F(x) := [-p(-x + x_0) + f(x_0), p(x + x_0) - f(x_0)].$$

One can easily see that all assumptions of Theorem 2 are fulfilled. Thus there exists a linear extension $g : X \to E$ of f such that

$$g(x) \in F(x) \quad \text{for } x \in C.$$

If $x \in x_0 + C$, then

$$g(x) = g(x_0) + g(x - x_0) = f(x_0) + g(x - x_0)$$
$$\leq f(x_0) + p(x - x_0 + x_0) - f(x_0) = p(x).$$

This completes the proof. ■

Our corollary gives Theorem 2.1 from [4] in the case $2x_0 - D = D$.

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ON THE CONVOLUTION EQUATION $\check{\mu} \star \rho \star \mu = \rho$

Introduction

It has been resently proved in [B] that for any probability measure μ on a countable (discrete) group G the existence of nontrivial (i.e. nonzero) solutions ρ of the convolution equation ($\diamond$) $\check{\mu} \star \rho \star \mu = \rho$ is equivalent to the concentration (see definition below) of the measure μ. By ${}_\star P_\star(\mu)$ we denote the convex set of all probabilities ρ on G which solve ($\diamond$). Our definitions and notation follow [B]. For the reader's convenience we briefly recall some of them. By the support of a measure μ on G we mean the set $S(\mu) = \{g \in G : \mu(g) > 0\}$. If $S(\mu)$ is finite we say that the measure μ is finitary. The convolution of measures μ, ν is defined

$$\mu \star \nu(g) = \sum_{h \in G} \mu(gh^{-1})\nu(h) = \sum_{h \in G} \mu(h)\nu(h^{-1}g). \tag{1}$$

Clearly $\mu \star \nu$ belongs to the set $P(G)$ of all probabilities on G if both μ and ν are from $P(G)$. Moreover $(P(G), \star)$ is an associative semigroup. It follows from (1) that $S(\mu \star \nu) = S(\mu)S(\nu)$. By $\check{\mu}$ we denote the symmetric reflection of a measure μ (i.e. $\check{\mu}(g) = \mu(g^{-1})$) and $\nu_1 \wedge \nu_2$ stands for the minimum of ν_1 and ν_2. For a fixed probability measure μ on G we define a positive linear operator P_μ acting on real (or complex) functions f on G

$$P_\mu f(g) = \sum_{h \in G} f(gh)\, \mu(h). \tag{2}$$

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It is well known that each $\mu \in P(G)$ defines a (right) random walk $\{\xi_n\}_{n\geq 0}$ on the group G. The transition probabilities are:

$$\text{(3)} \qquad \operatorname{Prob}(\xi_{n+1} = g | \xi_n = h) = \mu(h^{-1}g) \quad g, h \in G.$$

Thus for any natural n, $A \subseteq G$ and $h \in G$ we have

$$\text{(4)} \qquad \operatorname{Prob}(\xi_n \in A | \xi_0 = h) = \mu^{\star n}(h^{-1}A).$$

In this note we continue investigations, originated in [B], of the asymptotic behaviour of $\sup\{\mu^{\star n}(hA) : h \in G\}$, where A are finite subsets of G.

Definition 1. A concentration function of a probability measure $\mu \in P(G)$ is the set function $\mathbb{K}_\mu$ defined

$$\mathbb{K}_\mu(A) = \sup_{h \in G} \mu(hA).$$

We say that a measure $\mu \in P(G)$ is *concentrated* if there exist a finite set $A \subseteq G$ and a sequence $g_n \in G$ such that

$$\mathbb{K}_{\mu^{\star n}}(A) = \mu^{\star n}(g_n^{-1}A) \equiv 1.$$

We say that a measure $\mu \in P(G)$ is *not scattered* if there exists a finite set $A \subseteq G$ such that

$$\overline{\lim_{n \to \infty}} \mathbb{K}_{\mu^{\star n}}(A) > 0.$$

We say that a measure $\mu \in P(G)$ is *scattered* if for each finite set $A \subseteq G$ we have

$$\lim_{n \to \infty} \mathbb{K}_{\mu^{\star n}}(A) = 0.$$

Concentration functions of random walks have been investigated for almost forty years. Crucial papers for our considerations are [DL] and [B]. In the second paper it is proved that random walks are either concentrated or scattered. Moreover, (see the Theorem below) it is established that the classes of concentrated and non–scattered random walks coincide. Several conditions eqiuvalent to concentration are given there. In this note we add new eqiuvalent conditions in the case when the measure μ is adapted.

Definition 2. A probability measure μ on G is said to be *adapted* if the smallest subgroup $\mathfrak{G}(\mu)$ containing $S(\mu)$ is the whole group G. By $\mathfrak{H}(\mu)$ we denote the smallest normal subgroup H of $\mathfrak{G}(\mu)$ such that for all $g \in S(\mu)$ we have $S(\mu) \subseteq gH$.

It has been discovered in [B] that if μ is concentrated then

$$\mathfrak{H}(\mu) = \bigcup_{n=1}^{\infty} S(\check{\mu}^{\star n} \star \mu^{\star n}) = \bigcup_{n=1}^{\infty} (S(\check{\mu}^{\star n} \star \mu^{\star n}) \cup S(\mu^{\star n} \star \check{\mu}^{\star n}))$$

is a finite subgroup of G. It is even true that for n being large enough we have $\mathfrak{H}(\mu) = S(\check{\mu}^{\star n} \star \mu^{\star n})$. This yields $a^{-1}\mathfrak{H}(\mu)b = \mathfrak{H}(\mu)$ for all $a, b \in S(\mu)$. The above property of concentrated measures brings our attention to the following family of permutations of the group G. Let $\Phi_{x,y}(g) = xgy^{-1}$ where $x, y, g \in G$.

DEFINITION 3. Given a probability measure $\mu \in P(G)$ by $\mathcal{A}(\mu)$ we denote the group of 1–1 and onto transformations of G generated by all $\Phi_{a,b}$ where $a, b \in S(\mu)$. A set $D \subseteq G$ is called *forward-back* (shortly f–b) invariant if $\Phi(D) = D$ for all $\Phi \in \mathcal{A}(\mu)$. A f–b invariant set D is called *forward-back minimal* if there are no f–b invariant sets included in D other than D.

Since $\mathcal{A}(\mu)$ is a group thus there exist f–b minimal sets and all of them have the form $\{\Phi(g) : \Phi \in \mathcal{A}(\mu)\}$, for some $g \in G$. Let $\mathcal{D}(\mu)$ denote the partition of G onto f–b minimal sets.

DEFINITION 4. Given a probability measure $\mu \in P(G)$ by $\mathcal{A}_\dagger(\mu)$ we denote the group of 1–1 and onto transformations of G generated by all $\Phi_{x,y}$ such that $x = a_n^{\varepsilon_n} a_{n-1}^{\varepsilon_{n-1}} \dots a_1^{\varepsilon_1}$, $y = b_n^{\sigma_n} b_{n-1}^{\sigma_{n-1}} \dots b_1^{\sigma_1}$ where $a_j, b_j \in S(\mu)$, $\varepsilon_j, \sigma_j \in \{-1, 1\}$ and $\sum_{j=1}^{n} \varepsilon_j = \sum_{j=1}^{n} \sigma_j$. By $\mathcal{D}_\dagger(\mu)$ we denote the partition of G onto minimal sets defined by the group $\mathcal{A}_\dagger(\mu)$ and we call them f–b $\dagger$ minimal.

Since $\mathcal{A}(\mu) \subseteq \mathcal{A}_\dagger(\mu)$ thus the partition $\mathcal{D}_\dagger(\mu)$ should be finer. However for concentrated μ we find the partitions $\mathcal{D}(\mu), \mathcal{D}_\dagger(\mu)$ are same. The existence at least one finite set of the partition $\mathcal{D}(\mu)$ or $\mathcal{D}_\dagger(\mu)$ is equivalent to concentration of μ and this fact is the main point of our *Theorem 1*. The partitions $\mathcal{D}(\mu)$ or $\mathcal{D}_\dagger(\mu)$ are also used to describe the geometry of ${}_\star P_\star(\mu)$. We find out that either there is only one trivial solution $\rho = 0$ of $(\diamond)$ or if G is infinite and μ is adapted then the set ${}_\star P_\star(\mu)$ is infinite dimensional. Since solutions of $(\diamond)$ form a Banach sublattice of $\ell^1(G)$ thus in the second case ${}_\star P_\star(\mu)$ is an affine and isometric copy of $\{(t_n)_{n=1}^{\infty} : \sum_{n=1}^{\infty} t_n = 1,\ t_n \geq 0\}$ (so an affine and isometric copy of $P(G)$). We finish the introductory part with the following *Theorem* which comes from [B].

THEOREM (see [B]). *Let μ be a probability measure on a countable group G. Then the following conditions are equivalent:*

(i) *μ is concentrated*
(ii) *μ is not scattered*
(iii) *there exists a function $f \in \ell^2(G)$ such that* $\lim_{n\to\infty} \|P_\mu^n f\|_2 > 0$

(iv) *there exists a probability measure* ρ *on* G *such that* $\check{\mu} \star \rho \star \mu = \rho$

(v) $\lim_{n\to\infty} \operatorname{card}(S(\mu^{\star n})) < \infty$

(vi) $\mathfrak{H}(\mu)$ *is finite.*

Results

The above *Theorem* gives us a convenient tool in investigating of random walks on discrete groups. Let us notice that the question whether a probability measure μ on countable G is concentrated may be studied using computers. By [B] nonfinitary measures may be excluded since they are scattered. For a concrete discrete group G and a finitary measure $\mu \in P(G)$ we may build an algorithm describing the sequence $S(\check{\mu}^{\star n})S(\mu^{\star n}) = \mathfrak{H}_n(\mu)$. If at some moment $n+1$ the sets $\mathfrak{H}_{n+1}(\mu)$ and $\mathfrak{H}_n(\mu)$ coincide then the procedure may be stopped with the conclusion that μ is concentrated. Moreover in this case the group $\mathfrak{H}(\mu)$ is exactly $\mathfrak{H}_n(\mu)$. To prove *Theorem 1* we need

LEMMA 1. *If* μ *is adapted and concentrated then* $\mathcal{D}(\mu)$ *and* $\mathcal{D}_{\dagger}(\mu)$ *coincide with the family of cosets of* $\mathfrak{H}(\mu)$.

Proof. Since for concentrated μ the subgroup $\mathfrak{H}(\mu)$ may be represented as $\bigcup_{n=1}^{\infty} S(\mu)^{-n}S(\mu)^n = S(\mu)^{-n_\mu}S(\mu)^{n_\mu}$ for some natural n_μ thus $\mathfrak{H}(\mu)$ is f-b invariant set. It is also a normal subgroup, so for any $a, a_1, \ldots, a_n \in S(\mu)$ and $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$ we have $a_n^{\varepsilon_n} \cdots a_1^{\varepsilon_1}\mathfrak{H}(\mu) = a^{\sum_{j=1}^{n}\varepsilon_j}\mathfrak{H}(\mu)$. In particular $\mathfrak{H}(\mu)$ is f–b $_\dagger$ invariant. Clearly it is f–b minimal since $\mathfrak{H}(\mu) = \{\Phi(e) : \Phi \in \mathcal{A}(\mu)\}$ where e denotes the neutral element of G.This implies that the subgroup $\mathfrak{H}(\mu)$ is f–b $_\dagger$ minimal. Since μ is assumed to be adapted on the same way we may prove f–b or f–b $_\dagger$ minimality of any coset $g\mathfrak{H}(\mu)$. It follows that the partitions $\mathcal{D}(\mu)$ and $\mathcal{D}_\dagger(\mu)$ coincide with classes $\{g\mathfrak{H}(\mu)\}_{g\in G}$ and the proof of lemma is completed. ∎

Now we are in a position to prove:

THEOREM 1. *Let* μ *be an adapted measure on a countable group* G. *Then the following conditions are equivalent*:

(i) μ *is concentrated*

(vii) *there exists a finite f–b* $_\dagger$ *invariant set*

(viii) *there exists a finite f–b invariant set*

(ix) *all sets of the partition* $\mathcal{D}_\dagger(\mu)$ *are finite and coincide with classes of* $\mathfrak{H}(\mu)$

(x) *all sets of the partition* $\mathcal{D}(\mu)$ *are finite and coincide with classes of* $\mathfrak{H}(\mu)$.

Proof. Implications (vii)$\Longrightarrow$(viii), (ix)$\Longrightarrow$(x)$\Longrightarrow$(viii) and (ix)$\Longrightarrow$(vii) are obvious. (i)$\Longrightarrow$(ix) and (i)$\Longrightarrow$(x) follow *Lemma 1*. So we only have to prove that (viii) implies (i). To show this we prove that existence of a finite f-b invariant set $D \subseteq G$ implies nontrivial solutions of ($\diamond$). For this let us take τ_D, the uniform distribution on D. We have

$$\begin{aligned}\check{\mu} \star \tau_D \star \mu(g) &= \sum_{a,b \in S(\mu)} \tau_D(agb^{-1})\mu(a)\mu(b) \\ &= \begin{cases} 0 & \text{if } g \notin D \\ \sum\limits_{a,b \in S(\mu)} \frac{1}{\operatorname{card}(D)}\mu(a)\mu(b) & \text{otherwise} \end{cases} \\ &= \begin{cases} 0 & \text{if } g \notin D \\ \frac{1}{\operatorname{card}(D)} & \text{otherwise} \end{cases} = \tau_D(g).\end{aligned}$$

Thus $\tau_D \in {}_\star P_\star(\mu)$ and the proof of the *Theorem 1* is completed. ■

Remark 1. Notice that by the same arguments $\tau_D \in {}_\star P_\star(\check{\mu})$.

Given a probability measure μ on G let $\{\xi_n\}_{n=1}^\infty$ denote the random walk generated by μ and $\{\tilde{\xi}_n\}_{n=1}^\infty$ its independent copy. Consider a Markov process $\{\eta_n\}_{n=1}^\infty$ on G defined as

(5) $$\eta_n = \tilde{\xi}_n^{-1} \cdot \eta_0 \cdot \xi_n$$

where η_0 is independent of $\tilde{\xi}$ and ξ. If ρ_0 is a distribution of η_0 then the distribution of η_n is $\check{\mu}^{\star n} \star \rho_0 \star \mu^{\star n}$. For symmetric μ processes like (5) belong to the class of so called bilateral random walks. In this note we obtain a satisfactory description of their asymptotic distributions. We begin with:

THEOREM 2. *Let μ be an adapted probability measure on a countable group G. Then the convex set ${}_\star P_\star(\mu)$ i s either empty (if μ is scattered) or consists of all probabilities $\rho \in P(G)$ having the representation*

(6) $$\rho = \sum_{D \in \mathcal{D}(\mu)} \alpha_D \tau_D$$

(if μ is concentrated) where $\alpha_D \geq 0$, $\sum\limits_{D \in \mathcal{D}(\mu)} \alpha_D = 1$ and $\tau_D(\cdot) = \frac{\operatorname{card}(\cdot \cap D)}{\operatorname{card}(\mathfrak{H}(\mu))}$ is the uniform distribution on the set $D \in \mathcal{D}(\mu)$. Moreover extreme points of ${}_\star P_\star(\mu)$ coincide with measures τ_D.

Proof. We may assume that μ is concentrated. It is noticed in the proof of *Theorem 1* that $\tau_D \in {}_\star P_\star(\mu)$. First we prove (6). For $\rho \in_\star P_\star(\mu)$ we set $E_1 = \{g_1 \in G : \rho(g_1) = \alpha_1\}$ where $\alpha_1 = \sup\{\rho(g) : g \in G\}$. Clearly E_1 is

nonempty and finite. If $g_1 \in E_1$ then

$$\rho(g_1) = \check{\mu} \star \rho \star \mu(g_1) = \sum_{a,b\in S(\mu)} \rho(ag_1b^{-1})\mu(a)\mu(b),$$

so for all $a, b \in S(\mu)$ the points ag_1b^{-1} belong to E_1. It implies E_1 is f–b invariant, so may be decomposed on finite many sets of the partition $\mathcal{D}(\mu)$. Since ρ is uniformly distributed on E_1 thus $\rho|_{E_1} = \sum_{D\subseteq E_1} \alpha_D\tau_D$, where $\alpha_D \equiv \alpha_1 \mathrm{card}\,(\mathfrak{H}(\mu))$ does not depend on $D \subseteq E_1$.

Assume that there are pairwise disjoint sets $E_1, \ldots, E_{k-1}$ each of them is a finite union of elements of the partition $\mathcal{D}(\mu)$, and that for any $g_j \in E_j$ ($1 \le j \le k-1$) we have

$$\rho(g_j) = \sup_{g\in G\setminus(E_1\cup\ldots E_{j-1})} \rho(g) = \alpha_j.$$

This implies $\rho|_{E_1\cup\ldots E_{k-1}} = \sum_{D\subseteq E_1\cup\ldots E_{k-1}} \alpha_D\tau_D$, where $\alpha_D = \alpha_j \mathrm{card}(\mathfrak{H}(\mu))$ for $D \subseteq E_j$. Now we set $\alpha_k = \sup_{g\in G\setminus(E_1\cup\ldots\cup E_{k-1})} \rho(g) < \alpha_{k-1}$ and define $E_k = \{g \in G : \rho(g) = \alpha_k\}$. If $\alpha_k = 0$ our procedure may be stopped. If not, we show that again E_k is a finite union of f–b minimal sets disjoint from $E_1, \ldots, E_{k-1}$. Since the last sets are f–b invariant thus $\Phi(E_k) \subseteq G\setminus(E_1\cup\ldots\cup E_{k-1})$ for all $\Phi \in \mathcal{A}(\mu)$. On the other hand if $g_k \in E_k$ then we have $\rho(g_k) = \sum_{a,b\in S(\mu)} \rho(ag_kb^{-1})\mu(a)\mu(b)$ so $\rho(\Phi_{a,b}(g_k)) \equiv \alpha_k$. This implies $\Phi_{a,b}(E_k) \subseteq E_k$ for all $a, b \in S(\mu)$. Since the set E_k is finite and the transformations Φ are 1–1, thus E_k is f–b invariant. The same arguments as before lead us to the representation $\rho|_{E_k} = \sum_{D\subseteq E_k} \alpha_D\tau_D$ where $\alpha_D = \alpha_k \mathrm{card}(\mathfrak{H}(\mu))$. Now by induction the decomposition (6) is easily seen.

In order to prove that extreme points of ${}_\star P_\star(\mu)$ are exactly measures τ_D it is sufficient to show that the support $S(\rho)$ of extremal $\rho \in \mathrm{ex}_\star P_\star(\mu)$ is f–b minimal, and that two distinct extremal solutions of ($\diamond$) have disjoint supports. If $S(\rho)$ is not f–b minimal then by the decomposition (6) $\rho = \sum_{D\in\mathcal{A}(\mu)} \alpha_D\tau_D$ has at least two nonzero coefficients α_D. Clearly such ρ is not extremal. Now let $\rho_1 \neq \rho_2$ be from ex ${}_\star P_\star(\mu)$. Since

$$\check{\mu} \star (\rho_1 \wedge \rho_2) \star \mu \le (\check{\mu} \star \rho_1 \star \mu) \wedge (\check{\mu} \star \rho_2 \star \mu) = \rho_1 \wedge \rho_2$$

and

$$\sum_{g\in G} \check{\mu} \star (\rho_1 \wedge \rho_2) \star \mu(g) = \sum_{g\in G} \rho_1 \wedge \rho_2(g) = \beta$$

thus $\rho_1 \wedge \rho_2 = 0$ (it holds if $\beta = 0$ and then $S(\rho_1) \cap S(\rho_2) = \emptyset$) or $\beta > 0$ and

then $\frac{\rho_1 \wedge \rho_2}{\beta} \in_\star P_\star(\mu)$. Clearly $\beta < 1$ since $\rho_1 \neq \rho_2$. But $0 < \beta < 1$ gives

$$\rho_j = \beta \frac{\rho_1 \wedge \rho_2}{\beta} + (1-\beta) \frac{\rho_j - \rho_1 \wedge \rho_2}{1-\beta}.$$

For extreme ρ_1 and ρ_2 the above is possible only if $\rho_1 = \rho_2$. As a result we get that different extremal solutions ρ_1 and ρ_2 of ($\diamond$) satisfy $S(\rho_1) \cap S(\rho_2) = \emptyset$, and the proof of *Theorem 2* is completed. ■

COROLLARY 1. *Let μ be an adapted probability measure on a countable group G. If μ is concentrated then ${}_\star P_\star(\mu)$ is an affine and isometric copy of $\{(t_j)_{j=1}^N : \sum_{j=1}^N t_j = 1, t_j \geq 0\}$ where $N = \frac{\text{card } (G)}{\text{card } (\mathfrak{H}(\mu))}$ if G is finite and $N = \infty$ if G is infinite.*

COROLLARY 2. *For any adapted probability measure μ on a countable group G we have ${}_\star P_\star(\mu) =_\star P_\star(\check{\mu})$.*

Proof. By *Theorem* from [B] μ is concentrated if and only if $\check{\mu}$ is concentrated. Since $\mathcal{A}(\mu) = \mathcal{A}(\check{\mu})$ thus $\mathcal{D}(\mu) = \mathcal{D}(\check{\mu})$, so for concentrated μ the decomposition (6) gives ${}_\star P_\star(\mu) =_\star P_\star(\check{\mu})$. ■

Now we will study asymptotic behaviour of distributions of η_n. Obviously we may drop the case of scattered μ. In fact, for such measures we have:

$$\sup_{\nu \in P(G)} \sup_{g \in G} \check{\mu}^{\star n} \star \nu \star \mu^{\star n}(gA) =$$

$$\sup_{\nu \in P(G)} \sup_{g \in G} \sum_{a,c \in G} \mu^{\star n}(c^{-1}agA)\nu(c)\mu^{\star n}(a) \leq$$

$$\sup_{\nu \in P(G)} \sum_{a,c \in G} (\sup_{g \in G} \mu^{\star n}(gA))\nu(c)\mu^{\star n}(a) =$$

$$\sup_{g \in G} \mu^{\star n}(gA) \overset{n \to \infty}{\longrightarrow} 0$$

for any finite set $A \subseteq G$ and $\nu \in P(G)$. Thus the process η_n is scattered as well. On the other hand if μ is concentrated and adapted the situation is different. For any initial distribution ν the distribution of η_n becomes exponentially stationary. Namely, we have

THEOREM 3. *Let μ be an adapted and concentrated measure on a countable discrete group G. Then for some $C > 0$ and $\gamma > 0$ the following estimation*

(7)
$$\sup_{\nu \in P(G)} \|\check{\mu}^{\star n} \star \nu \star \mu^{\star n} - \sum_{D \in \mathcal{D}(\mu)} \nu(D)\tau_D\| \leq Ce^{-\gamma n}$$

holds, where $\|\cdot\|$ stands for $\ell^1(G)$ (or equivalently variation) norm.

Proof. For an element $D = g\mathfrak{H}(\mu)$ of the partition $\mathcal{D}(\mu)$ let $\{\eta_{n,D}\}_{n\geq 0}$ denote the restriction of the Markov chain to the f–b invariant set D. Now $\{\eta_{n,D}\}_{n\geq 0}$ is a finite Markov chain with transition probabilities

$$p^D_{g_1,g_2} = \sum_{a^{-1}g_1b=g_2} \mu(a)\mu(b).$$

It is proved in *Theorem 1* that the measure τ_D is a stationary probabilistic vector for this transition matrix. Let h $_\mu$ = card (D) = card $(\mathfrak{H}(\mu))$. With this notation $[p^D_{g_1,g_2}]_{h_\mu\times h_\mu}$ is a $h_\mu \times h_\mu$ doubly stochastic. We show that the family $\{P^{(D)}\}_{D\in\mathcal{D}(\mu)}$ is "uniformly irreducible". Let us fix for a while $D \in \mathcal{D}(\mu)$. For any $g \in D$ the states which may be reached in time n starting from g are exactly $S(\check{\mu}^{\star n})gS(\mu^{\star n}) = D_{g,n} \subseteq D$. Let $g = a_m^{\varepsilon_m}\ldots a_1^{\varepsilon_1}$ for some $a_m,\ldots,a_1 \in S(\mu)$ and $\varepsilon_m,\ldots,\varepsilon_1 \in \{-1,1\}$. By the same arguments as in the proof of *Theorem* in [B] we get

$$S(\check{\mu}^{\star n})gS(\mu^{\star n}) = S(\check{\mu}^{\star n})S(\mu^{(n+\sum_{j=1}^m \varepsilon_j)}) = \mathfrak{H}(\mu)g = D$$

if n is large enough. Thus $D_{g,n} = D$ for all $n \geq L(g)$.Let $L(D) = \sup_{g\in D} L(g)$ and $L = \sup_{D\in\mathcal{D}(\mu)} L(D)$.We show that $L < \infty$. Firstly we notice that for any fixed $D \in \mathcal{D}(\mu)$ and arbitrary $g \in D$ if $D_{g,n} = D_{g,k}$ for some $k > n$ then $D_{g,n} = D$. In fact, for any $j > n$ we have

$$D_{g,j} = S(\check{\mu}^{\star(j-n)})D_{g,n}S(\mu^{\star(j-n)}),$$

so $D_{g,n} = D_{g,k}$ implies that the sequence $\{D_{g,j}\}_{j\geq n}$ is periodic. Since for large j the sets $D_{g,j}$ are stabilized as D, by periodicity for some $n \leq j \leq k$ we have $D_{g,j} = D$. This means that $D = D_{g,j} = D_{g,k} = D_{g,n}$. By our *Lemma 1* all sets D of the partition $\mathcal{D}(\mu)$ have exactly h $_\mu$ elements. Obviously there are only finite many 1–1 sequences of subsets of a finite set D, and the amount of all such sequences is a function of h $_\mu$. It follows $L < \infty$.

In particular we get that for all $D \in \mathcal{D}(\mu)$ and $g_1, g_2 \in D$ the transition probabilities $p^{D,n}_{g_1,g_2} = \sum_{a^{-1}g_1b=g_2} \mu^{\star n}(a)\mu^{\star n}(b)$ are strictly positive if $n \geq L$. This implies that for some α we have $p^{D,n}_{g_1,g_2} \geq \alpha = \inf_{g\in S(\mu^{\star n})}(\mu^{\star n}(g))^2 > 0$. So the matrix of transition probabilities at the time n satisfies

$$(P^{(D)})^n = [p^{D,n}_{g_i,g_j}]_{h_\mu\times h_\mu} \geq \alpha[1]_{h_\mu\times h_\mu}$$

where $[1]_{h_\mu\times h_\mu}$ denotes the $h_\mu \times h_\mu$ matrix with 1's as entries. From the theory of doubly stochastic matrices $(P^{(D)})^n$ convereges to $[1/h_\mu]_{h_\mu\times h_\mu}$

(with exponential rate). Thus for some $C > 0$ and $\gamma > 0$ the inequality $\|(P^{(D)})^n - [1/h_\mu]_{h_\mu \times h_\mu}\| \leq Ce^{-\gamma n}$ holds. The constants C and γ depend on α and h μ but not on any particular $D \in \mathcal{D}(\mu)$. As a result we get the estimation

$$\sup_{D\in\mathcal{D}(\mu)} \sup_{A\subseteq D} \sup_{\nu\in P(D)} |\check{\mu}^{\star n} \star \nu \star \mu^{\star n}(A) - \tau_D(A)| \leq Ce^{-\gamma n},$$

where $P(D)$ denotes the set of all probabilities ν such that $S(\nu) \subseteq D$.

Let $\nu \in P(G)$ be an initial distribution of the Markov chain $\{\eta_n\}_{n\geq 0}$ and $A \subseteq G$ be arbitrary. By ν_D we denote the conditional probability of ν on D (i.e. $\nu_D(\cdot) = \frac{\nu(\cdot\cap D)}{\nu(D)}$) if $\nu(D) > 0$ or something if $\nu(D) = 0$. Now we have

$$\Big|\mathrm{Prob}_\nu(\eta_n \in A) - \sum_{D\in\mathcal{D}(\mu)} \nu(D)\tau_D(A)\Big| =$$

$$\Big|\sum_{D\in\mathcal{D}(\mu)} (\mathrm{Prob}_\nu(\eta_n \in D\cap A) - \nu(D)\tau_D(A))\Big| \leq$$

$$\sum_{D\in\mathcal{D}(\mu)} \nu(D)|\mathrm{Prob}_{\nu_D}(\eta_{n,D} \in D\cap A) - \tau_D(D\cap A)| \leq$$

$$\sum_{D\in\mathcal{D}(\mu)} \nu(D)Ce^{-\gamma n} = Ce^{-\gamma n}.$$

Here $A \subseteq G$ and $\nu \in P(G)$ are arbitrary so we obtain (7) and the proof of *Theorem 3* is completed. ■

Corollary 3. *Let μ be an adapted probability measure on countable G and T_μ denote the stochastic operator on the Banach lattice $\ell^1(G)$ defined as $T_\mu(\nu) = \check{\mu} \star \nu \star \mu$. Then the following two conditions are equivalent:*

(i) *μ is concentrated*

(xi) *there exist a stochastic projection Q_μ and constants $C > 0$, $\gamma > 0$ such that*

$$\|T_\mu^n - Q_\mu\|_{\mathrm{oper}} \leq Ce^{-\gamma n}$$

where $\|\cdot\|_{oper}$ is the operator norm on $\mathcal{L}(\ell^1(G))$.

Moreover if the above hold then $Q_\mu = \sum_{D\in\mathcal{D}(\mu)} \lambda_D \otimes \tau_D$ where $\lambda_D(\nu) = \nu(D)$.

Proof. Any finite signed measure ν on G may be represented as $\nu = s\nu_1 + t\nu_2$ where ν_1, ν_2 are orthogonal probabilities and $\|\nu\| = |s| + |t|$. ■

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REMARK ON GLOBALLY LIPSCHITZIAN COMPOSITION OPERATORS

Introduction

Let $I \subseteq \mathbb{R}$ be an interval, $f : I \times \mathbb{R} \to \mathbb{R}$ a fixed two-place function, and $\mathcal{F}(I)$ the linear space of all the functions $u : I \to \mathbb{R}$. The function $F : \mathcal{F}(I) \to \mathcal{F}(I)$ given by the formula

$$(F(u))(x) := f(x, u(x)), \quad x \in I, \ u \in \mathcal{F}(I),$$

is said to be a composition operator.

Let $a \in I$ be fixed. Denote by Lip(I) the Banach space of all the functions $u \in \mathcal{F}(I)$ with the norm

$$\|u\|_{\mathrm{Lip(I)}} := |u(a)| + \sup\left\{\frac{u(x_1) - u(x_2)}{x_1 - x_2} : x_1, x_2 \in I;\ x_1 \neq x_2\right\}. \tag{1}$$

In [2] it is proved that if a composition operator F mapping Lip(I) into itself is globally Lipschitzian with respect to the Lip(I)-norm, then $f(x,y) = g(x)y + h(x)$, $(x \in I; y \in \mathbb{R})$, for some $g, h \in$Lip(I).

Next this result has been extended to some other function Banach spaces (cf. [1] for references). In particular, (cf. [3]) if F is a globally Lipschitzian selfmap of $\mathbf{C}_n(I)$, i.e. there is an $L \geq 0$ such that

$$\|F(u) - F(v)\|_{C_n(I)} \leq L\|u - v\|_{C_n(I)}, \quad u, v \in \mathbf{C}_n(I), \tag{2}$$

where

$$\|u\|_{C_n(I)} := \sum_{i=0}^{n-1} |u^{(i)}(a)| + \sup\{|u^{(n)}(x)| : x \in I\}, \tag{3}$$

then $f(x,y) = g(x)y + h(x)$, $(x \in I; y \in \mathbb{R})$, for some $g, h \in \mathbf{C}_n(I)$.

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In the present note we generalize these results. We show that the basic assumption of the global Lipschitz continuity of the composition operator can be essentially weakened. It turns out that the main result of [3] remains valid if the inequality (2) holds only for u, v being the polynomials at most of the degree n. Moreover, the argument presented here is much simpler than that in [3] where some complicated chain rule formulas are used.

1. Main result

We start with the following

Remark 1. Let $x_1, x_2, u_1, u_2 \in \mathbb{R}$, $x_1 \neq x_2$, and $n \in \mathbb{N}$, be arbitrarily fixed. Then it is easy to verify that the polynomial $u : \mathbb{R} \to \mathbb{R}$ given by

$$u(x) := a_n(x-a)^n + a_1 x + a_0,$$

where

$$a_n = \frac{u_1 - u_2}{k!(x_1 - x_2)}, \qquad a_1 = \frac{u_1 - u_2}{x_1 - x_2}\left(1 - \frac{(x_1 - a)^n - (x_2 - a)^n}{n!(x_1 - x_2)}\right),$$

$$a_0 = u_1 - \frac{u_1 - u_2}{n!(x_1 - x_2)}(x_1 - a)^n - \frac{u_1 - u_2}{x_1 - x_2}\left(1 - \frac{(x_1 - a)^n - (x_2 - a)^n}{n!(x_1 - x_2)}\right)x_1$$

has the following properties

$$u(x_1) = u_1, \quad u(x_2) = u_2; \qquad \|u\|_{C_n[a,b]} = |a_0| + |a_1| + \left|\frac{u_1 - u_2}{x_1 - x_2}\right|.$$

In the same way, taking arbitrary $v_1, v_2 \in \mathbb{R}$, we can find a polynomial

$$v(x) := b_n(x-a)^n + b_1 x + b_0$$

such that

$$v(x_1) = v_1, \quad v(x_2) = v_2; \qquad \|v\|_{C_n[a,b]} = |b_0| + |b_1| + \left|\frac{v_1 - v_2}{x_1 - x_2}\right|.$$

where b_n, b_1, b_0 are defined as a_n, a_1, a_0 with u_1, u_2 replaced by v_1, v_2.

From the formulas for a_0, a_1, b_0, b_1 it easy to observe that

$$\|u - v\|_{C_n[a,b]} = |a_0 - b_0| + |a_1 - b_1| + \left|\frac{u_1 - u_2 - v_1 + v_2}{x_1 - x_2}\right|,$$

and, if x_1 and x_2 tend to an $x \in \mathbb{R}$, then there exists a $c(x) \in \mathbb{R}$ such that

$$\lim_{x_1, x_2 \to x} |x_1 - x_2| \|u - v\|_{C_n(I)} = c(x)|u_1 - u_2 - v_1 + v_2|. \tag{4}$$

Denote by $\mathbf{P}_n(I)$ the set of all the real polynomials of the degree at most n, restricted to the interval I.

THEOREM. *Let $F : \mathcal{F}(I) \to \mathcal{F}(I)$ be the composition operator generated by a function $f : I \times \mathbb{R} \to \mathbb{R}$, and suppose that $n, m \in \mathbb{N}$ are fixed positive integers. If F maps $\mathbf{P}_n(I)$ into $\mathbf{C}_m(I)$ and there exists an $L \geq 0$ such that*

$$\|F(u) - F(v)\|_{C_m(I)} \leq L\|u - v\|_{C_n(I)}, \quad u, v \in \mathbf{P}_n(I), \tag{5}$$

then there exist $g, h \in \mathbf{C}_m(I)$ such that

$$f(x, y) = g(x)y + h(x), \quad x \in I, \ y \in \mathbb{R}.$$

Proof. We have $F(u) = f(\cdot, y)$ for each constant function $u(x) := y \in \mathbb{R}$. Since F maps $\mathbf{P}_n(I)$ into $\mathbf{C}_m(I)$, it follows that, for every $y \in \mathbb{R}$, the function $f(\cdot, y)$ is continuous in I.

From the definition of the norms (1) and (3) we get

$$\|u\|_{\mathrm{Lip(I)}} \leq \|u\|_{C_k(I)}, \quad u \in \mathbf{C}_k(I), \ k \in \mathbb{N},$$

and inequality (5) implies

$$\|F(u) - F(v)\|_{\mathrm{Lip(I)}} \leq L\|u - v\|_{C_n(I)}, \quad u, v \in \mathbf{P}_n(I). \tag{6}$$

Let us fix arbitrary $x_1, x_2 \in I, x_1 \neq x_2$; $u_1, u_2, v_1, v_2 \in \mathbb{R}$, and take the polynomials u and v constructed in Remark 1. Making use of the definition of Lip(I)-norm and substituting u and v to the inequality (6), we obtain

$$\left|\frac{f(x_1, u_1) - f(x_2, u_2) - f(x_1, v_1) + f(x_2, v_2)}{x_1 - x_2}\right| =$$

$$\left|\frac{f(x_1, u(x_1)) - f(x_2, u(x_2)) - f(x_1, v(x_1)) + f(x_2, v(x_2))}{x_1 - x_2}\right| \leq$$

$$\leq \|F(u) - F(v)\|_{\mathrm{Lip(I)}} \leq L\|u - v\|_{C_n(I)},$$

which implies that

$$|f(x_1, u_1) - f(x_2, u_2) - f(x_1, v_1) + f(x_2, v_2)| \leq L|x_1 - x_2|\|u - v\|_{C_n(I)}.$$

Hence, letting x_1 and x_2 tend to an arbitrary fixed $x \in I$, and making use of (4) and the continuity of $f(\cdot, y)$ for every $y \in \mathbb{R}$, we obtain

$$|f(x, u_1) - f(x, u_2) - f(x, v_1) + f(x, v_2)| \leq Lc(x)|u_1 - u_2 - v_1 + v_2|, \tag{7}$$
$$x \in I, \ u_1, u_2, v_1, v_2 \in \mathbb{R}.$$

Substituting here $u_1 := y + w, u_2 := y, v_1 := w, v_2 := 0$, we get

$$f(x, y + w) - f(x, 0) = [f(x, y) - f(x, 0)] + [f(x, w) - f(x, 0)], \quad y, w \in \mathbb{R}.$$

Put $h(x) := f(x, 0)$, $x \in I$. It follows that, for each fixed $x \in I$, the function $\alpha_x : \mathbb{R} \to \mathbb{R}$ given by

$$\alpha_x(y) := f(x, y) - h(x), \quad y \in \mathbb{R}, \tag{8}$$

satisfies the Cauchy functional equation

$$\alpha_x(y + w) = \alpha_x(y) + \alpha_x(w), \quad y, w \in \mathbb{R}.$$

Taking $v_1 = v_2 := 0$ in (7) we get

$$|\alpha_x(u_1) - \alpha_x(u_2)| \leq L\, c(x)|u_1 - u_2|, \quad u_1, u_2 \in \mathbb{R}.$$

Hence, for each $x \in I$, α_x is additive and continuous. Consequently, for each $x \in I$, there exists a $g(x) \in \mathbb{R}$ such that $\alpha_x(y) = g(x)y$, $y \in \mathbb{R}$. Now from (8) we have

$$f(x, y) - h(x) = g(x)y, \quad x \in I,\ y \in \mathbb{R}.$$

Since $h(x) = f(x, 0) = F(0)$ and $g(x) = f(x, 1) - f(x, 0) = F(1) - F(0)$, $x \in I$, we have $g, h \in \mathbf{C}_m(I)$. This completes the proof.

Remark 2. It is easy to observe that the above Theorem remains true on replacing the norm $\|\cdot\|_{C_n(I)}$ in (6) by any norm $\|\cdot\|$ such that for some $M > 0$ and all $u \in \mathbf{P}_n(I)$, we have $\|u\| \leq M\|u\|_{C_n(I)}$.

2. Some Corollaries

As an immediate consequence of Theorem we obtain

COROLLARY 1. *Let $f : I \times \mathbb{R} \to \mathbb{R}$ and let n, m be positive integer such that $m \leq n$. The composition operator F generated by f maps the space $\mathbf{C}_n(I)$ into $\mathbf{C}_m(I)$ and is globally Lipschitzian, i.e. there exists an $L > 0$ such that*

$$\|F(u) - F(v)\|_{C_m(I)} \leq L\|u - v\|_{C_n(I)}, \quad u, v \in \mathbf{C}_n(I),$$

if and only if there exist $g, h \in \mathbf{C}_m(I)$ such that

$$f(x, y) = g(x)y + h(x), \quad x \in I, y \in \mathbb{R}.$$

Remark 3. If in the above Corollary we have $m > n$, then $f(x, y) = h(x)$, $x \in I$, $y \in \mathbb{R}$ (cf. [3], also [1], p. 211, Theorem 8.3).

Because $\|u\|_{C_1(I)} = \|u\|_{\mathrm{Lip(I)}}$ for all $u \in \mathbf{P}_1(I)$, by an obvious change in the proof of Theorem 1, we obtain the following generalization of the result proved in [2] and quoted in the Introduction.

COROLLARY 2. *Let $f : I \times \mathbb{R} \to \mathbb{R}$. If the composition operator F generated by f maps $\mathbf{P}_1(I)$ into* Lip (I) *and there exists an $L \geq 0$ such that*

$$\|F(u) - F(v)\|_{\mathrm{Lip(I)}} \leq L\|u - v\|_{\mathrm{Lip(I)]}}, \qquad u, v \in \mathbf{P}_1(I),$$

then there exist $g, h \in$ Lip (I) *such that*

$$f(x, y) = g(x)y + h(x), \qquad x \in I,\ y \in \mathbb{R}.$$

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A NEW PROOF OF THE ORTHOGONALITY OF JACOBI POLYNOMIALS

The Jacobi polynomials constitute an important and a rather wide class of orthogonal polynomials, from which Chebyshev, Legendre, Laguerre and Gegenbauer polynomials follow as special cases ([1], pp. 189–190). It is well known that the moment problem for Jacobi polynomials is positive-definite, hence their weight function is unique and the interval of orthogonality is finite. However, their orthogonality, with the nonnegative weight function $(1-x)^{\alpha}(1+x)^{\beta}$ on the interval $[-1,1]$, for $\alpha > -1, \beta > -1$, is usually derived by the use of the associated differential equation and Rodrigue's formula. In this paper, we employ the Saalschütz's theorem to obtain a new, much simpler, and perhaps more elegant, method of establishing their orthogonality.

THEOREM. *For $\alpha > -1$, $\beta > -1$ Jacobi polynomials are orthogonal with the weight function $(1-x)^{\alpha}(1+x)^{\beta}$ on $[-1,1]$ according as*

$$(1)\qquad \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} P_n^{(\alpha,\beta)}(x) P_k^{(\alpha,\beta)}(x)\,dx = 0 \qquad \text{for } k \neq n,$$

$$= \frac{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{n!(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+n+1)} \qquad \text{for } k = n.$$

Proof. We express the Jacobi polynomials in terms of hypergeometric function ([3], p. 268) as

$$P_n^{(\alpha,\beta)}(x) = \binom{n+\alpha}{n} {}_2F_1(-n, n+\alpha+\beta+1; \alpha+1; (1-x)/2)$$

and the integral in (1) can be written as

$$(2) \qquad \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} \frac{(\alpha+1)_n}{n!} \frac{(\alpha+1)_k}{k!} \times$$
$$\times {}_2F_1(-n, n+\alpha+\beta+1; \alpha+1; (1-x)/2) \times$$
$$\times {}_2F_1(-k, k+\alpha+\beta+1; \alpha+1; (1-x)/2)\, dx.$$

By the definition ([1], p. 322, (10.1)) of the hypergeometric function

$${}_2F_1(-n, n+\alpha+\beta+1; \alpha+1; (1-x)/2) =$$
$$= \sum_{u=0}^{\infty} \frac{(-n)_u (n+\alpha+\beta+1)_u}{(\alpha+1)_u u!} \left(\frac{1-x}{2}\right)^u,$$

(2) becomes

$$(3) \qquad \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} \frac{(\alpha+1)_n}{n!} \sum_{u=0}^{\infty} \frac{(-n)_u (n+\alpha+\beta+1)_u}{(\alpha+1)_u u! 2^u} (1-x)^u \times$$
$$\times \frac{(\alpha+1)_k}{k!} \sum_{m=0}^{\infty} \frac{(-k)_m (k+\alpha+\beta+1)_m}{(\alpha+1)_m m! 2^m} (1-x)^m\, dx.$$

Interchanging the order of the integral and the summations, which is justified because of their absolute convergence, (3) can be rewritten as

$$(4) \qquad \frac{(\alpha+1)_n}{n!} \frac{(\alpha+1)_k}{k!} \sum_{m=0}^{\infty} \frac{(-k)_m (k+\alpha+\beta+1)_m}{(\alpha+1)_m m! 2^m} \times$$
$$\times \sum_{u=0}^{\infty} \frac{(-n)_u (n+\alpha+\beta+1)_u}{(\alpha+1)_u u! 2^u} \int_{-1}^{1} (1-x)^{\alpha+m+u}(1+x)^{\beta}\, dx.$$

We now use the standard result that, for $\alpha > -1, \beta > -1$,

$$\int_{-1}^{1} (1-y)^{\alpha}(1+y)^{\beta} dy = 2^{\alpha+\beta+1} \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)},$$

and (4) becomes

$$(5) \quad 2^{\alpha+\beta+1}\frac{(\alpha+1)_n}{n!}\frac{(\alpha+1)_k}{k!}\Gamma(\beta+1)\sum_{m=0}^{\infty}\frac{(-k)_m(k+\alpha+\beta+1)_m}{(\alpha+1)_m m!}\times$$

$$\times\sum_{u=0}^{\infty}\frac{(-n)_u(n+\alpha+\beta+1)_u\Gamma(\alpha+m+u+1)}{(\alpha+1)_u u!\Gamma(\alpha+m+u+\beta+2)}.$$

The second summation in (5) can now be replaced by the generalized hypergeometric function to enable us to rewrite (5) as

$$(6) \quad 2^{\alpha+\beta+1}\frac{(\alpha+1)_n}{n!}\frac{(\alpha+1)_k}{k!}\Gamma(\beta+1)\times$$

$$\times\sum_{m=0}^{\infty}\frac{(-k)_m(k+\alpha+\beta+1)_m\Gamma(\alpha+m+1)}{(\alpha+1)_m m!\Gamma(\alpha+\beta+m+2)}\times$$

$$\times{}_3F_2(-n,n+\alpha+\beta+1,\alpha+m+1;\alpha+1,\alpha+\beta+m+2).$$

It can easily be verified that the generalized hypergeometric series in (6) is Saalschützian. Therefore, by the Saalschutz's theorem ([2], p. 188, (3)), (6) becomes

$$(7) \quad 2^{\alpha+\beta+1}\frac{(\alpha+1)_k}{n!k!}\Gamma(\beta+1)\times$$

$$\times\sum_{m=0}^{\infty}\frac{(-k)_m(k+\alpha+\beta+1)_m\Gamma(\alpha+m+1)(-n-\beta)_n(-m)_n}{(\alpha+1)_m m!\Gamma(\alpha+\beta+m+2)(-n-\alpha-\beta-m-1)_n}.$$

Since $(-k)_m = 0$ for $m > k$ and $(-m)_n = 0$ for $n > m$, it is noted that all terms in the series in (7) are zero for $k < n$, and hence the integral (1) is zero.

For $k = n$ we use the standard result $(-n)_n = (-1)^n n!$ and obtain from (7)

$$\frac{2^{\alpha+\beta+1}\Gamma(\beta+1)\Gamma(\alpha+n+1)\Gamma(\alpha+\beta+2n+1)(-n-\beta)_n}{n!\Gamma(\alpha+\beta+n+2)\Gamma(\alpha+\beta+n+1)(-\alpha-\beta-2n-1)_n}$$

which, on using [2] (p. 3, (3), (4)) and some algebraic manipulations, reduces to

$$\frac{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{n!(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+n+1)}.$$

This completes the proof of the Theorem.

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GENERALIZED SOLUTIONS OF A GOURSAT-TYPE PROBLEM FOR THE POLYWAVE EQUATION IN $\mathbb{R}^n$-SPACE

Introduction

Generalized solutions of Goursat-type problems in $\mathbb{R}^2$-space were defined and studied in papers [1], [2]. Similar results concerning the $\mathbb{R}^3$- space were obtained in Chapter II of the unpublished paper [5] (which was based on [4]), and for another Goursat problem in [3]*). In this paper, which contains the results of [5], we examine generalized solutions of a Goursat-type problem in $\mathbb{R}^n$-space where n is an arbitrary positive integer not less than three. Our argument is based on papers [7] and [8].

1. The problem and assumptions

Let Ω be the parallelepiped

$$\Omega = \{x \in \mathbb{R}^n : 0 \leq x \leq A\}$$

($x = (x_s)$, where $s = 1, 2, \ldots, n$) and Y a Banach space with the norm $\|\cdot\|$.

In what follows N denotes the set of all positive integers.

For fixed $p \in N$, we consider the polywave (or polyvibrating) equation of Mangeron (cf [6])

(1.1)
$$L^p u(x) = F(x)$$

($x \in \Omega$), where $L = \prod_{\mu=1}^n$ with $D_\mu = \frac{\partial}{\partial x_\mu}$, $L^k = L(L^{k-1})$ for $k = 1, 2, \ldots, p$; $L^0 u = u$, and F is a given function.

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*) Concerning the classical solutions of Goursat-type problems in $\mathbb{R}^n$, where $n \geq 3$, see [4], [8] and the references therein

By a solution of equation (1.1) in Ω we mean a function $u : \Omega \to Y$ such that (cf [8])

$$D_{i_1} \dots D_{i_l} L^k u \in C^{p-k-1} \quad \text{for } k = 0, 1, \dots, p-1$$

$(1 \le i_1 < \dots < i_l \le n;\ l = 1, 2, \dots, n)$ satisfying (1.1) for $x \in \Omega$.

Let $x^{(i)} = (x_s^{(i)})$, where $x_s^{(i)} = x_s$ for $1 \le s \le i-1$ $(2 \le i \le n)$; $x_s^{(i)} = x_{s+1}$ for $i \le s \le n-1$ $(1 \le i \le n-1)$, denote by Ω_i the set of all points $x^{(i)}$ for $x \in \Omega$ (of course $\Omega_i = \mathop{\times}\limits_{\substack{s=1\\ s\neq i}}^{n} [0, A_s]$), and consider a system of surfaces $S_1, \dots, S_n$ given by the equations

$$x_i = f_i(x^{(i)})$$

$(x^{(i)} \in \Omega_i)$, respectively, where $f_i : \Omega_i \to [0, A_i]$ for $i = 1, 2, \dots, n$.

We examine the Goursat-type problem $(\mathfrak{G})$ that consists in finding a solution of equation (1.1) in Ω, subject to the boundary conditions

$$L^r u(x) = N_{i,r}(x^{(i)}) \quad \text{for } x \in S_i \tag{1.2}$$

$(x^{(i)} \in \Omega_i;\ i = 1, 2, \dots, n;\ r = 0, 1, \dots, p-1)$, where $N_{i,r} : \Omega_i \to Y$ are given functions.

Each function having the said properties is called a classical solution (briefly c.s.) of the $(\mathfrak{G})$-problem.

Now, we are going to define generalized solutions (briefly g.s.) of the $(\mathfrak{G})$-problem (our definition originates from those in [1], [2]).

To this end let us consider a sequence $\{(\mathfrak{G}^m)\}$ (where $m \in N$; $m > m_0$ with m_0 being a sufficiently large positive integer) of Goursat problems which are formulated analogously to $(\mathfrak{G})$ with the replacement of F, $N_{i,r}$ and S_i by F^m, $N_{i,r}^m$ and S_i^m, respectively (S_i^m denotes a surface of equation $x_i = f_i^m(x^{(i)})$, where

$$F^m : \Omega \to Y,\ N_{i,r}^m : \Omega_i \to Y \text{ and } f_i^m : \Omega_i \to [0, A_i]$$

$(i = 1, 2, \dots, n;\ r = 0, 1, \dots, p-1)$ are given functions.

We admit the following definition

DEFINITION 1.1 A function $u : \Omega \to Y$ is called a g.s. of the $(\mathfrak{G})$-problem if there is a sequence $\{u^m\}$ of functions $u^m : \Omega \to Y$ $(m \in N;\ m > m_0)$ such that

1° Each of the functions u^m is a c.s. of the corresponding Goursat problem $(\mathfrak{G}^m)$ in which the given functions satisfy the relations

$$F^m \rightrightarrows F;\ f_i^m \rightrightarrows f_i;\ N_{i,r}^m \rightrightarrows N_{i,r} \quad \text{when } m \to \infty \tag{1.3}$$

($i = 1, 2, \ldots, n$; $r = 0, 1, 2, \ldots, p-1$ and $\rightrightarrows$ denotes the uniform convergence), and

2° The following relation

$$u^m \rightrightarrows u \quad \text{when } m \to \infty \tag{1.4}$$

holds good.

We make the following assumptions:

I. The functions $f_i : \Omega_i \to [0, A_i]$ ($i = 1, 2, \ldots, n$) are Hölder-continuous (exponent $h_f \in (0, 1]$), the surfaces S_i ($i = 1, 2, \ldots, n$) do not intersect one another at the points of Ω placed outside the axes of coordinates and the following inequality is satisfied

$$f_i(x^{(i)}) \le K_1[\min_{1 \le s \le n-1} x_s^{(i)}]^{n-1} \tag{1.5}$$

($i = 1, 2, \ldots, n$), where K_1 is a positive constant such that

$$\vartheta := K_1 A^{n-2} < 1 \tag{1.6}$$

with $A = \max_{1 \le i \le n} A_i$.

II. The functions $N_{i,r} : \Omega_i \to Y$ ($i = 1, 2, \ldots, n$; $r = 0, 1, \ldots, p-1$) are Hölder-continuous (exponent $h_N \in (0, 1]$) and satisfy the inequality

$$\|N_{i,r}(x^{(i)})\| \le K_2[\min_{1 \le s \le n-1} x_s^{(i)}]^{c_r} \tag{1.7}$$

($i = 1, 2, \ldots, n$; $r = 0, 1, \ldots, p-1$), where K_2 is a positive constant and $c_r = n + p - r - 1$.

III. The function F is continuous.

2. Auxiliary theorems

Set $\vec{k}(n) = (k_v)$, where $v = 1, 2, \ldots, n$; $v \ne i$; $x^{(i)}_{\vec{k}^i(n),m} = x^{(i)}$ with $x_s^{(i)} = A_s \frac{k_s}{m}$ for $s = 1, 2, \ldots, n$, $s \ne i$;

$$w_{\vec{k}^i(n)}(x^{(i)}) = \prod_{v=i}^{n} \binom{m}{k_v} x_v^{k_v}(A_v - x_v)^{m-k_v}; \quad B_i = \prod_{v=1}^{n} A_v \tag{2.1}$$

($v \ne i$), and consider the Bernstein polynomials

$$f_i^m(x^{(i)}) = B_i^{-m} \sum_{k_v=n-1}^{m} f_i(x^{(i)}_{\vec{k}^i_{(n)},m}) w_{\vec{k}^i_{(n)}}(x^{(i)}) \tag{2.2}$$

($v = 1, 2, \ldots, n$; $v \ne i$), where $i = 1, 2, \ldots, n$; $m \in N$; $m \ge n-1$.

LEMMA 2.1. *The following relations*

$$0 \le f_i^m(x^{(i)}) \le A_i; \quad f_i^m \in C^\infty(\Omega_i); \tag{2.3}$$

$$f_i^m \rightrightarrows f_i \quad \textit{when } m \to \infty; \tag{2.4}$$

$$D^l f_i^m(x^{(i)}) = 0 \quad \textit{when } \prod_{s=1}^{n-1} x_s^{(i)} = 0; \quad 0 \le |l| \le n-2 \tag{2.5}$$

hold good, where

$$D^l = \prod_{v=1}^{n} D_v^{l_v}; \quad |l| = \sum_{v=1}^{n} l_v \quad (v \ne i). \tag{2.5'}$$

Proof. Relations (2.3) and (2.5) follow immediately from (2.1) and (2.2). In order to prove (2.4), let us observe that by (2.2) we can write

$$|f_i^m(x^{(i)}) - f_i(x^{(i)})| = \left| f_i^m(x^{(i)}) - f_i(x^{(i)}) B_i^{-m} \sum_{k_v=0}^{m} w_{\vec{k}^i(n)}(x^{(i)}) \right| \le \tag{2.6}$$

$$\le B_i^{-m} \Big\{ \sum_{k_v=0}^{m} |f_i(x^{(i)}) - f_i(x^{(i)}_{\vec{k}^i(n),m})| w_{\vec{k}^i(n)}(x^{(i)}) +$$

$$+ \sum_{t=0}^{n-1} \sum_{k_1,\dots,k_t=0}^{m} \sum_{k_{t+1}=0}^{n-2} \sum_{k_{t+2},\dots,k_n=n-1}^{m} f_i(x^{(i)}_{\vec{k}^i(n),m}) w_{\vec{k}^i(n)}(x^{(i)}) \Big\}$$

$(k_{s_1}, k_{s_1+1},\dots, k_{s_2} = \emptyset$ for $s_1 > s_2)$.

Denote the terms on the right-hand side of (2.6) by $e_1^m(x^{(i)})$ and $e_2^m(x^{(i)})$, successively, and let $\varepsilon > 0$ be arbitrarily fixed.

It is well known (cf [9], p. 152) that there is a number $m_\varepsilon^* \in N$ such that

$$e_1^m(x^{(i)}) < \frac{\varepsilon}{2} \tag{2.7}$$

when $m > m_\varepsilon^*$.

For the term $e_2^m(x^{(i)})$ we have (cf. (1.5))

$$e_2^m(x^{(i)}) \le K_1 \sum_{s=1}^{n} \sum_{k_s=0}^{n-2} \binom{m}{k_s} \left(A_s \frac{k_s}{m}\right)^{n-1} x_s^{k_s}(A_s - x_s)^{m-k_s} \le \tag{2.8}$$

$$\le K_1(n-2)^{n-1} \sum_{s=1}^{n} A_s^{n-1} m^{1-n} \le K_1(n-1)^n A^{n-1} m^{1-n}$$

$(s \ne i)$, where $A = \max_{1 \le i \le n} A_i$, and as a consequence we can assert that there is a number $\widetilde{m}_\varepsilon \in N$ such that

$$e_2^m(x^{(i)}) < \frac{\varepsilon}{2} \tag{2.9}$$

when $m > \widetilde{m}_\varepsilon$.

On joining (2.6), (2.7) and (2.9) we get relation (2.4). Q.E.D.

LEMMA 2.2. *The surfaces S_i^m, of equations $x_i = f_i^m(x^{(i)})$, respectively $(i = 1, 2, \dots, n;\ m \in N;\ m \geq n-1)$ satisfy the following relation*

$$S_k \cap S_l = \{x \in \Omega : x_k = x_l = 0\}$$

$(k, l = 1, 2, \dots, n;\ k \neq l)$.

Proof. Suppose that S_k and S_l $(k \neq l)$ intersect at a point $\mathring{x} = (\mathring{x}_s) \in \Omega$ where $0 < \mathring{x}_k \leq A_k$ or $0 < \mathring{x}_l \leq A_l$. Then

$$\mathring{x}_k = f_k^m(\mathring{x}^{(k)}|_{\mathring{x}_l = f_l^m(\mathring{x}^{(l)})}) \quad \text{and} \quad \mathring{x}_l = f_l^m(\mathring{x}^{(l)}|_{\mathring{x}_k = f_k^m(\mathring{x}^{(k)})}). \tag{2.10}$$

We are going to prove that

$$f_k^m(x^{(k)}|_{x_l = f_l^m(x^{(l)})}) < x_k \tag{2.10'}$$

when $0 < x_k \leq A_k$.

To this end let us observe that formula (2.2) and Assumption I yield

$$f_i^m(x^{(i)}) \leq K_1 B_i^{-m} m^{1-n} \sum_{k_v=1}^{m} \prod_{s=1}^{n} A_s k_s \binom{m}{k_s} x_s^{k_s} (A_s - x_s)^{m-k_s}$$

$(v, s \neq i)$, whence we get

$$f_i^m(x^{(i)}) \leq K_1 m^{1-n} \prod_{s=1}^{n} A_s \sum_{k_s=1}^{m} k_s \binom{m}{k_s} \left(\frac{x_s}{A_s}\right)^{k_s} (1 - x_s)^{m-k_s}$$

$(s \neq i)$, and using the well known equality (cf [9], p. 150).

$$\alpha = m_\beta^{-1} \sum_{\beta=1}^{m} \beta \binom{m}{\beta} \alpha^\beta (1-\alpha)^{m-\beta}$$

we have

$$f_i^m(x^{(i)}) \leq K_1 \prod_{s=1}^{n-1} x_s^{(i)} \quad (i = 1, 2, \dots, n). \tag{2.11}$$

Basing on (1.5), (1.6) and (2.11), we obtain

$$f_k^m(x^{(k)}|_{x_l = f_l^m(x^{(l)})}) \leq K_1 \prod_{s=1}^{n-1} x_s^{(k)} f_l(x^{(l)}) \leq K_1^2 \prod_{s=1}^{n-1} x_s^{(k)} \prod_{r=1}^{n-1} x_r^{(l)} \leq$$
$$\leq (K_1 A^{n-2})^2 x_k < x_k$$

$(s \neq l;\ 0 < x_k \leq A_k)$, as required.

It is clear that inequality (2.10′) contradicts relations (2.10) and so Lemma 2.2 is valid. Q.E.D.

We have the following corollary whose validity follows from (1.5), (1.6) and (2.11).

COROLLARY 2.1. *The inequality*

$$\max(f_i^m(x^{(i)}),\ f_i(x^{(i)})) \leq \vartheta \min_{1\leq s\leq n-1} x_s^{(i)} \tag{2.12}$$

$(i = 1, 2, \ldots, n)$ *holds good.*

Now, let us consider the expressions $a_{r,j}^{i,m} : \Omega_i \to \mathbb{R}$ given by the formulae (cf. [7], [8])

$$a_{r,j}^{i,m}(x^{(i)}) = \begin{cases} x_r^{(j)} & \text{for } r \neq i \\ f_i^m(x^{(i)}) & \text{for } r = i \end{cases} \tag{2.13}$$

when $i < j$;

$$a_{r,j}^{i,m}(x^{(i)}) = \begin{cases} x_r^{(j)} & \text{for } r \neq i-1 \\ f_i^m(x^{(i)}) & \text{for } r = i-1 \end{cases} \tag{2.14}$$

when $i > j(x^{(i)} \in \Omega_i;\ 1 \leq i, j \leq n;\ r = 1, 2, \ldots, n-1)$, and the sequences $(z_{\vec{k}(t)}^{v,m})$ and $(u_{\vec{k}(t),j}^{v,m})$ defined by

$$z_{\vec{k}(t)}^{v,m}(x^{(v)}) = (z_{s,\vec{k}(t)}^{v,m}(x^{(v)})) \tag{2.15}$$

$(s = 1, 2, \ldots, n-1)$, where

$$z_{s,\vec{k}(t)}^{v,m}(x^{(v)}) = a_{s,k_t}^{k_{t-1},m}(z_{\vec{k}(t-1)}^{v,m}(x^{(v)}))$$

for $t = 2, 3, \ldots,$; $s = 1, 2, \ldots, n-1$

$$z_{s,\vec{k}(1)}^{v,m}(x^{(v)}) = a_{s,k_1}^{v,m}(x^{(v)}) \quad \text{for } s = 1, 2, \ldots, n-1 \tag{2.16}$$

$(\vec{k}(t) = (k_l)$ where $l = 1, 2, \ldots, t$; $t \in N$; $1 \leq k_l \leq n$; $k_l \neq k_{l-1}$; $k_0 = v$; $v = 1, 2, \ldots, n)$;

$$u_{\vec{k}(t),j}^{v,m}(x^{(v)}) = (u_{s,\vec{k}(t),j}^{v,m}(x^{(v)})) \tag{2.17}$$

$(s = 1, 2, \ldots, n-1)$, where

$$u_{s,\vec{k}(t),j}^{v,m}(x^{(v)}) = a_{s,j}^{k_t,m}(z_{\vec{k}(t)}^{v,m}(x^{(v)})) \quad \text{for } t = 1, 2, \ldots, \tag{2.18}$$

($\vec{k}(t)$ is understood as in (2.16), $k_t \neq j$; $j = 1, 2, \ldots, n$; $s = 1, 2, \ldots, n-1$; $v = 1, 2, \ldots, n$).

It is easily observed that

$$z_{\vec{k}(t)}^{v,m}(x^{(v)}) = u_{\vec{k}(t-1),k_t}^{v,m}(x^{(v)}) \tag{2.19}$$

$(v = 1, 2, \ldots, n;\ t = 2, 3, \ldots)$.

LEMMA 2.3. *For each number $\eta > 0$ there is a positive integer $m_* = m_*(\eta)$ such that the inequalities*

$$\max_{1\le v\le n}\max_{1\le s\le n-1}\sup_{\Omega_v}|z^{v,m}_{s,\vec{k}(t)}(x^{(v)}) - z^{v}_{s,\vec{k}(t)}(x^{(v)})| < t\eta;$$

$$(2.20)\qquad \max_{1\le v\le n}\max_{1\le s\le n-1}\max_{1\le j\le n}\sup_{\Omega_v}|u^{v,m}_{s,\vec{k}(t),j}(x^{(v)}) - u^{v}_{s,\vec{k}(t),j}(x^{(v)})| < t\eta$$

(*where* $z^v_{s,\vec{k}(t)}$ *and* $u^v_{s,\vec{k}(t),j}$ *are given by formulae analogous to* (2.16), (2.18), *respectively, with m being omitted*) *hold good for $t \in N$ and $m \in N$; $m > m_*(\eta)$.*

Proof of Lemma 2.3 is similar to that of Lemma 7 in [2].

Now, let us consider the following truncated Bernstein polynomials (cf. (1.7) and (2.1))

$$(2.21)\qquad N^m_{i,r}(x^{(i)}) = B_i^{-m}\sum_{k_v=c_r}^{m} N_{i,r}(x^{(i)}_{\vec{k}^i(n),m})w_{\vec{k}^i(n)}(x^{(i)})$$

($v = 1,2,\dots,n$; $v \neq i$), where $i = 1,2,\dots,n$; $r = 0,1,\dots,p-1$; $m \in N$; $m \ge n+p$.

LEMMA 2.4. *The following relations hold good*

$$(2.22)\qquad N^m_{i,r} : \Omega_i \to Y;\qquad N^m_i \in C^\infty(\Omega_i);$$

$$(2.23)\qquad N^m_{i,r} \rightrightarrows N_{i,r}\quad \text{when } m \to \infty;$$

$$(2.24)\qquad D^l N^m_{i,r}(x^{(i)}) = 0\quad \text{when } \prod_{s=1}^{n-1} x^{(i)}_s = 0;\qquad 0 \le |l| \le n-r+p-1$$

(D^l *is understood as in* (2.5′));

$$(2.25)\qquad |||N^{m(l)}_{i,r}(x^{(i)})|||_l \le C(m)\prod_{s=1}^{n-1} x^{(i)}_s$$

when $l = n+p-r-2$, $C(m)$ being a positive constant dependent on m. Above, $|||\cdot|||_l$ denotes the norm in the space of l-linear continuous functions from $\mathbb{R}^{n-1}$ into Y.

Proof. The proof of (2.23) is analogous to that of (2.4), and (2.24) follows from (2.1) and (2.21). It is also clear that $N^m_{i,r} \in C^\infty(\Omega_i)$. Thus, it suffices to prove (2.25). To this end let us observe that by (1.7) an (2.21) we have (cf. (2.5′))

$$\|D^l N^m_{i,r}(x^{(i)})\| \le \text{const}\, B_i^{-m}\sum_{k_v=c_r}^{m}\prod_{s=1}^{n}\binom{m}{k_s}\Big[\min\Big(A_v\frac{k_v}{m}\Big)\Big]^{c_r}\times$$

$$\times \sum_{\alpha_s=0}^{\tilde{m}_s} \binom{l_s}{\alpha_s} \frac{k_s!}{(k_s-l_s+\alpha_s)!} \frac{(m-k_s)!}{(m-k_s-\alpha_s)!} x_s^{k_s-l_s+\alpha_s}(A_s-x_s)^{m-k_s-\alpha_s}$$

$(v, s \neq i;\ \tilde{m}_s = \min(l_s, m-k_s)$ and $|l| = n+p-r-2)$, whence

$\|D^l N_{i,r}^m(x^{(i)})\| \leq$

$$\leq \widetilde{C}(m) \prod_{s=1}^{n} x_s^{c_r-l_s} \sum_{k_s=c_r}^{m} \sum_{\alpha_s=0}^{\tilde{m}_s} \binom{m}{k_s} \left(\frac{x_s}{A_s}\right)^{k_s-c_r+\alpha_s} \left(1-\frac{x_s}{A_s}\right)^{m-k_s-\alpha_s}$$

($s \neq i$ and $\widetilde{C}(m)$ is a positive constant dependent on m), and as a consequence we obtain

$$|||N_{i,r}^{m(l)}(x^{(i)})|||_l \leq C(m) \prod_{s=1}^{n-1} x_s^{(i)},$$

as required.

LEMMA 2.5. *The following inequality is valid*

$$\|N_{i,r}^m(x^{(i)})\| \leq K_2 C_* [\min_{1\leq s\leq n-1} x_s^{(i)}]^{c_r} \tag{2.26}$$

where $C_* = (c_r)^{c_r-1}$.

Proof. By (1.7) and (2.21) we can write

$$\|N_{i,r}^m(x^{(i)})\| \leq K_2 B_i^{-m} \sum_{k_v=c_r}^{m} \prod_{s=1}^{n} \binom{m}{k_s} \left[\min\left(A_v \frac{k_v}{m}\right)\right]^{c_r} x_s^{k_s}(A_s-x_s)^{m-k_s}$$

$(v, s \neq i)$, whence

$$\|N_{i,r}^m(x^{(i)})\| \leq K_2 m^{-c_r} A_{v_0}^{c_r} \sum_{k_{v_0}=c_r}^{m} \binom{m}{k_{v_0}} k_{v_0}^{c_r} \left(\frac{x_{v_0}}{A_{v_0}}\right)^{k_{v_0}} \left(1-\frac{x_{v_0}}{A_{v_0}}\right)^{m-k_{v_0}}$$

where v_0 is an arbitrarily fixed positive integer such that $1 \leq v_0 \leq n$; $v_0 \neq i$.

Now, it suffices to repeat the argument used in paper [2], p. 636 (with the replacement of $2p$ by c_r) to obtain the inequality

$$\|N_{i,r}^m(x^{(i)})\| \leq K_2 C_* x_{v_0}^{c_r}. \tag{2.27}$$

As v_0 $(1 \leq v_0 \leq n;\ v_0 \neq i)$ has been arbitrarily fixed, (2.27) yields the thesis (2.26). Q.E.D.

We shall end this section with the examination of the Bernstein polynomials

$$F^m(x) = B^{-m} \sum_{k_v=0}^{m} F(x_{\vec{k}(n),m}) \widetilde{w}_{\vec{k}(n)}(x) \tag{2.28}$$

$(v = 2, 3, \ldots, n)$ where $\vec{k}(n) = (k_s)$ with $s = 1, 2, \ldots, n$; $x_{\vec{k}(n),m} = x$ with $x_s = A_s \frac{k_s}{m}$;

$$\widetilde{w}_{\vec{k}(n)}(x) = \prod_{s=1}^{n} \binom{m}{k_s} x_s^{k_s} (A_s - k_s)^{m-k_s}; \quad B = \prod_{s=1}^{n} A_s. \tag{2.29}$$

It is evident that $F^m \in C^\infty(\Omega)$, and well known that

$$F^m \rightrightarrows F \quad \text{when } m \to \infty, \tag{2.30}$$

as a consequence of which the function

$$R_p^m(x) = [(p-1)!]^{-n} \int_0^{x_1} \ldots \int_0^{x_n} \prod_{v=1}^{n} (x_v - \eta_v)^{p-1} F(\eta)\, d\eta \tag{2.31}$$

tends uniformly in Ω to the limit

$$R_p(x) = [(p-1)!]^{-n} \int_0^{x_1} \ldots \int_0^{x_n} \prod_{v=1}^{n} (x_v - \eta_v)^{p-1} F(\eta)\, d\eta \tag{2.32}$$

when m tends to infinity.

3. The $(\mathfrak{G}^m)$-problems

It follows from the results of Section 2 (cf. Lemmas 2.1, 2.2, 2.4 and 2.5, and the properties of F^m) that the functions f_i^m, $N_{i,r}^m$ and F^m given by (2.2), (2.21) and (2.28), respectively satisfy the assumptions of paper [8] (cf. [8]], pp. 492, 493), and so, for each $(\mathfrak{G}^m)$-problem, i.e. the $(\mathfrak{G})$-problem generated by the said functions f_i^m, $N_{i,r}^m$ and F^m where $m > m_0$ with $m_0 \in N$ being sufficiently large, Theorem 2 of [8] concerning the existence of c.s. of this problem can be applied.

According to the said theorem, for each $m \in N$, $m > m_0$ the corresponding $(\mathfrak{G}^m)$-problem has a c.s. given by the formula

$$u^m(x) = R_p^m(x) + \sum_{j=1}^{p} \sum_{i=1}^{n} (x_i)^{p-j} \psi_{i,p-j}^m(x^{(i)}) \tag{3.1}$$

$(x \in \Omega)$, where

$$\psi_{i,p-j}^m(x^{(i)}) = \{(p-j)![(p-j-1)!]^{n-1}\}^{-1}. \tag{3.2}$$

$$\cdot \int_0^{x_1^{(i)}} \ldots \int_0^{x_{n-1}^{(i)}} \prod_{s=1}^{n-1} (x_s^{(i)} - \eta_s^{(i)})^{p-j-1} \phi_{i,p-j}^m(\eta_1^{(i)}, \ldots, \eta_{n-1}^{(i)})\, d\eta_1^{(i)} \ldots d\eta_{n-1}^{(i)}$$

for $j = 1,2,\ldots,p-1$; $\psi^m_{i,0} = \phi^m_{i,0}$ with $\phi^m_{i,p-j}$ defined by

$$\phi^m_{i,p-j}(x^{(i)}) = W^m_{i,p-j}(x^{(i)}) + \sum_{t=1}^{\alpha} V^{t,m}_{i,p-j}(x^{(i)}) \tag{3.3}$$

$(i = 1,2,\ldots,n;\ j = 1,2,\ldots,p)$. Above,

$$W^m_{i,p-j}(x^{(i)}) = N^m_{i,p-j}(x^{(i)}) - \overset{*m}{R}_{i,j}(x^{(i)}); \tag{3.4}$$

$$V^{t,m}_{i,p-j}(x^{(i)}) = (-1)^t \sum_{\vec{k}(t)} W^m_{k_t,p-j[z^{i,m}_{\vec{k}(t)}(x^{(i)})]}; \tag{3.5}$$

$$\overset{*m}{R}_{i,j}(x^{(i)}) = \overset{*m}{R}_{i,j}(x)|_{x_i = f_i(x^{(i)})} \tag{3.6}$$

with*)

$$\overset{*}{R}{}^m_j(x) = R^m_j(x) + \sum_{r=1}^{j-1} \{(j-r)![(j-r-1)!]^{n-1}\}^{-1} \sum_{k=1}^{n} (x_k)^{j-r}. \tag{3.7}$$

$$\cdot \int_0^{x_1^{(k)}} \ldots \int_0^{x_{n-1}^{(k)}} \prod_{s=1}^{n-1} (x_s^{(k)} - \eta_s^{(k)})^{j-r-1} \phi_{k,p-r}(\eta^{(k)})\, d\eta_1^{(k)} \ldots d\eta_{n-1}^{(k)}.$$

This solution is unique in the set of all solutions of equation (1.1) (with F replaced by F^m) in Ω, which (cf. Lemma 1 in [8]) are given by formula (3.1), such that the functions $\phi^m_{i,p-j}$ $(i = 1,2,\ldots,n;\ j = 1,2,\ldots,p)$ appearing in (3.2) satisfy the condition

$$|||\phi^{m(l)}_{i,p-j}(x^{(i)})|||_l \le C(\min_{1\le s\le n-1} x_s^{(i)})^{j+n-r-1}$$

(C is a positive constant depending in general on $\phi^m_{i,p-j}$) for $l = 0,1,\ldots,j+n-2$.

4. Generalized solutions of the ($\mathfrak{G}$)-problem

We shall prove the following theorem

THEOREM 4.1. *If Assumptions I–III are satisfied, then there is a g.s. of the ($\mathfrak{G}$)-problem given by the following formula*

$$u(x) = R_p(x) + \sum_{j=1}^{p}\sum_{i=1}^{n} x_i^{p-j} \psi_{i,p-j}(x^{(i)}) \tag{4.1}$$

---

*) The functions $R^m_j(x)$ $(j = 1,2,\ldots,p)$ are given by formula (2.31) with p replaced by j.

($x \in \Omega$; $x^{(i)} \in \Omega_i$; $i = 1, 2, \ldots, n$) *in which* $R_p(x)$ *is defined by* (2.32) *and the functions* $\psi_{i,p-j}$ ($i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, p$) *are given by the relations* (3.2)–(3.7) *with* m *being omitted.*

Proof. Let $N \ni m > m_0$, $m_0 \in N$ being sufficiently large, and consider the sequences of functions $\{f_i^m\}$, $\{N_{i,r}^m\}$ and $\{F^m\}$ given by (2.2), (2.21) and (2.28), respectively, and the sequnce of Goursat problems $\{(\mathfrak{G}^m)\}$ generated by these functions (i.e. such that, for each $N \ni m > m_0$, the said functions f_i^m, $N_{i,r}^m$ and F^m are the given functions appearing in $(\mathfrak{G}^m)$).

We know from Lemmas 2.1 and 2.4, and formula (2.28), that the aforesaid functions f_i^m, $N_{i,r}^m$ and F^m ($i = 1, 2, \ldots, n$; $r = 0, 1, \ldots,$) satisfy relations (1.3), respectively.

We also know from Section 3 that each of the $(\mathfrak{G}^m)$-problems has a solution u^m given by formula (3.1) together with (3.2)–(3.7).

Thus, in order to prove Theorem 4.1 it is sufficient to show that relation (1.4) is satisfied, where u^m and u are given by (3.1) and (4.1), respectively.

Let $\varepsilon > 0$ be a given positive number and observe that (cf. (3.1) and (4.1))

$$\|u^m(x) - u(x)\| \le E_1^m(x) + E_2^m(x) \tag{4.2}$$

where

$$E_1^m(x) = [(p-1)!]^{-n} \int_0^{x_1} \ldots \int_0^{x_n} \prod_{r=1}^{n} (x_r - \eta_r)^{p-1} \|F^m(\eta) - F(\eta)\| \, d\eta; \tag{4.3}$$

$$E_2^m(x) = \sum_{j=1}^{p} \sum_{i=1}^{n} x_i^{p-j} \|\psi_{i,p-j}^m(x^{(i)}) - \psi_{i,p-j}(x^{(i)})\| \tag{4.4}$$

($x \in \Omega$).

It is evident (cf. (2.30)) that there is a number $N \ni m_1 = m_1(\varepsilon) > m_0$ such that

$$E_1^m(x) < \frac{\varepsilon}{2} \tag{4.5}$$

for $x \in \Omega$; $N \ni m > m_1$.

In order to estimate the expression $E_2^m(x)$ we apply the method of mathematical induction.

Set $j = 1$. In this case (cf. (3.4)–(3.7))

$$W_{i,p-1}^m(x^{(i)}) = N_{i,p-1}^m(x^{(i)}) - R_1^m(x)|_{x_i = f_i^m(x^{(i)})}; \tag{4.6}$$
$$W_{i,p-1}(x^{(i)}) = N_{i,p-1}(x^{(i)}) - R_1(x)|_{x_i = f_i(x^{(i)})}.$$

Let $\theta \in (0,1)$. Basing on (4.6), and using Assumptions I–III and inequality (2.11), we get

$$(4.7) \qquad \|W^m_{i,p-1}(x^{(i)}) - W_{i,p-1}(x^{(i)})\| \le \text{const}\, \rho^\theta_m \big(\min_{1\le s\le n-1} x^{(i)}_s\big)^{2(1-\theta)}$$

where ρ_m is given by

$$(4.8) \qquad \rho_m = \max_{1\le i\le n} \max\{\sup_{\Omega} \|F^m(x) - F(x)\|,\ \sup_{\Omega_i} |f^m_i(x^{(i)}) - f_i(x^{(i)})|,$$
$$\sup_{\Omega_i} \|N^m_{i,p-1}(x^{(i)}) - N_{i,p-1}(x^{(i)})\|\}.$$

Let us observe that, by (4.6) and Assumptions I–III, we have

$$(4.9) \quad W_{i,p-1}(x^{(i)}) - W_{i,p-1}(\overline{x}^{(i)})\| \le \text{const}\big[\max_{1\le s\le n-1} |\overline{x}^{(i)}_s - x^{(i)}_s|\big]^{h_*\cdot\theta}\cdot$$
$$\cdot\big[\max\big(\min_{1\le s\le n-1} \overline{x}^{(i)}_s,\ \min_{1\le s\le n-1} x^{(i)}_s\big]^{1-\theta}$$

($h_* = \min(h_f, h_N)$), where $x^{(i)}, \overline{x}^{(i)} \in \Omega_i$; $i = 1, 2, \ldots, n$.

Using (4.7) and (4.9), we get

$$\|W^m_{k_t,p-1}(z^{i,m}_{\vec{k}(t)}(x^{(i)})) - W_{k_t,p-1}(z^{i}_{\vec{k}(t)}(x^{(i)}))\| \le$$
$$\le \|W^m_{k_t,p-1}(z^{i,m}_{\vec{k}(t)}(x^{(i)})) - W_{k_t,p-1}(z^{i,m}_{\vec{k}(t)}(x^{(i)}))\| +$$
$$+ \|W^m_{k_t,p-1}(z^{i,m}_{\vec{k}(t)}(x^{(i)})) - W_{k_t,p-1}(z^{i}_{\vec{k}(t)}(x^{(i)}))\| \le$$
$$\le \text{const}\{\rho^\theta_m[\max_{\vec{k}(t)} \min_{1\le r\le n-1} z^{i,m}_{r,\vec{k}(t)}(x^{(i)})]^{2(1-\theta)} + [\max_{\vec{k}(t)} \max_{1\le i\le n-1} \max_{1\le r\le n-1}$$
$$\sup_{\Omega_i} |z^{i,m}_{r,\vec{k}(t)}(x^{(i)}) - z^{i}_{r,\vec{k}(t)}(x^{(i)})|]^{k_*\cdot\theta}\cdot$$
$$\cdot[\max_{\vec{k}(t)} \max(\min_{1\le r\le n-1} z^{i,m}_{r,\vec{k}(t)}(x^{(i)}),\ \min_{1\le r\le n-1} z^{i}_{r,\vec{k}(t)}(x^{(i)}))]^{1-\theta}\},$$

whence, and by inequality (2.12), Corollary 1 in [7] and Lemma 2.3 above, we obtain

$$(4.10) \quad \|W^m_{k_t,p-1}(z^{i,m}_{\vec{k}(t)}(x^{(i)})) - W_{k_t,p-1}(z^{i}_{\vec{k}(t)}(x^{(i)}))\| \le \text{const}(\vartheta)^{t(1-\theta)} t \frac{\varepsilon}{\varkappa_0}$$

($\varkappa_0$ is a positive integer to be chosen later — cf. (4.19)), on condition that $N \ni m > m^{(1)} = m^{(1)}(\varepsilon, \varkappa_0)$.

As $\vartheta \in (0,1)$, formulas (3.5) and (4.10) yield

$$\sum_{t=1}^{\infty} \|V_{i,p-1}^{t,m}(x^{(i)}) - V_{i,p-1}^{t}(x^{(i)})\| \leq \text{const}\, \frac{\varepsilon}{\varkappa_0} \tag{4.11}$$

and as a consequence of (3.3), (4.7) and (4.11) we obtain the inequality

$$\|\phi_{i,p-1}^{m}(x^{(i)}) - \phi_{i,p-1}(x^{(i)})\| \leq C_1 \frac{\varepsilon}{\varkappa_0} \tag{4.12}$$

whence (cf. (3.2))

$$\|\psi_{i,p-1}^{m}(x^{(i)}) - \psi_{i,p-1}(x^{(i)})\| \leq \widetilde{C}_1 \frac{\varepsilon}{\varkappa_0} \tag{4.13}$$

($N \ni m > m^{(1)}$), C_1 and $\widetilde{C}_1$ being positive constant.

Now, let $j_0 \in N$ be arbitrarily fixed so that $1 \leq j_0 \leq p-1$ and assume that

$$\|\phi_{i,p-j}^{m}(x^{(i)}) - \phi_{i,p-j}(x^{(i)})\| \leq C_j \frac{\varepsilon}{\varkappa_0} \tag{4.14}$$

for $j = 1,2,\ldots,j_0$ when $N \ni m > \max_{1\leq v\leq j_0} m^{(v)}$, $m^{(v)} = m^{(v)}(\varepsilon, \varkappa_0)$ ($v = 1,2,\ldots,j_0$) being sufficiently large positive integers and $C_1,\ldots,C_{j_0}$ positive constants.

Evidently (cf. relation (3.2) satisfied by the functions $\psi_{i,p-j}^{m}$ and $\psi_{i,p-j}$), the said assumption yields

$$\|\psi_{i,p-j}^{m}(x^{(i)}) - \psi_{i,p-j}(x^{(i)})\| \leq \widetilde{C}_j \frac{\varepsilon}{\varkappa_0} \tag{4.15}$$

(m as above, $j = 1,2,\ldots,j_0$), where $\widetilde{C}_1,\ldots,\widetilde{C}_{j_0}$ are positive constants.

Basing on (3.3)–(3.7) and (4.14), and using an argument similar to that in the proof of (4.12), we get

$$\|\phi_{i,p-(j_0+1)}^{m}(x^{(i)}) - \phi_{i,p-(j_0+1)}(x^{(i)})\| \leq C_{j_0+1} \frac{\varepsilon}{\varkappa_0} \tag{4.16}$$

when

$$N \ni m > \max_{1\leq v\leq j_0+1} m^{(v)} m^{(j_0+1)} = m^{(j_0+1)}(\varepsilon, \varkappa_0)$$

is a sufficiently large positive integer), whence and by (3.2) we obtain

$$\|\psi_{i,p-(j_0+1)}^{m}(x^{(i)}) - \psi_{i,p-(j_0+1)}(x^{(i)})\| \leq \widetilde{C}_{j_0+1} \frac{\varepsilon}{\varkappa_0} \tag{4.17}$$

(C_{j_0+1} and $\widetilde{C}_{j_0+1}$ are positive constant).

Thus, by (4.13), (4.15), (4.17) and the induction principle, we can assert that the inequality (4.15) holds good for $j = 1, 2, \ldots, p$ when $N \ni m > \max_{1 \le v \le p} m^{(v)} (m^{(v)} = m^{(v)}(\varepsilon, \varkappa_0); v = 1, 2, \ldots, p$ are sufficiently large positive integers, with $\widetilde{C}_1, \ldots, \widetilde{C}_p$ being positive constants.

As a consequence of the aforesaid result and equality (4.4), we have

$$E_2^m(x) \le \widehat{C} \frac{\varepsilon}{\varkappa_0} \tag{4.18}$$

(where $\widehat{C}$ is a positive constant) when $N \ni m > \widetilde{m}_2 = m_2(\varepsilon, \varkappa_0)$.

Choosing $\varkappa_0 \in N$ so that $\frac{\hat{C}}{\varkappa_0} < \frac{1}{2}$, we can conclude that there is a number $N \ni m_2 = m_2(\varepsilon) > m_0$ such that

$$E_2^m(x) < \frac{\varepsilon}{2} \tag{4.19}$$

for $x \in \Omega$; $N \ni m > m_2$.

Inequalities (4.2), (4.5) and (4.19) yield

$$\|u^m(x) - u(x)\| < \varepsilon \tag{4.20}$$

for $x \in \Omega$; $N \ni m > \max(m_1, m_2)$, which ends the proof of Theorem 4.1.

Now, we shall prove the following theorem

THEOREM 4.2. *Let us assume that for $N \ni m > m_0$ (cf. the proof of Theorem 4.1) the following conditions concerning the $(\mathfrak{G}^m)$-problems are fulfilled*

1°. *The functions $f_i^m : \Omega_i \to [0, A_i]$ $(i = 1, 2, \ldots, n)$ have the properties expressed by Lemmas 2.1–2.5;*

2°. *The functions $N_{i,r}^m : \Omega_i \to Y$ $(i = 1, 2, \ldots, n; r = 0, 1, \ldots, p-1)$ have the properties expressed by Lemmas 2.4, 2.5 (with C_* in (2.26) replaced by any positive constant independent of m);*

3°. *The functions $F^m : \Omega \to Y$ have the properties mentioned on p. 11 and are equibounded together with their first-order partial derivatives;*

4°. *The functions $\psi_{i,p-j}^m$ $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, p)$ appearing in formula (3.1) for c.s. of the $(\mathfrak{G}^m)$-problems satisfy the inequality*

$$\|\psi_{i,p-j}^{m(v)}(x^{(i)})\| \le C_v (\min_{1 \le s \le n-1} x_s^{(i)})^{n+p-v-1} \tag{4.21}$$

$(i = 1, 2, \ldots, n; j = 1, 2, \ldots, p; v = 0, 1, \ldots, p+n-2)$, where C_v are positive constants independent of m.

Then, there is at most one g.s. of the $(\mathfrak{G})$-problem

Proof. It is our aim to show that if $\{f_i^{\mu,m}\}$, $\{N_{i,r}^{\mu,m}\}$, $\{F^{\mu,m}\}$ and $\{u_\mu^m\}$ $(\mu = 1,2,)$ satisfy the conditions of Definition 1.1 and the assumptions of Theorem 4.2, then the corresponding generalized solutions u_μ $(\mu = 1.2)$ of the $(\mathfrak{G})$-problem are identical in Ω.

To this end let us observe that

$$\|u_2(x) - u_1(x)\| \leq \|u_2(x) - u_2^m(x)\| + \|u_1(x) - u_1^m(x)+ + \|u_2^m(x) - u_1^m(x)\| \tag{4.22}$$

$(x \in \Omega)$ and that, for an arbitrary $\varepsilon > 0$ there is a sufficiently large number $N \ni m_0' = m_0'(\varepsilon)$ such that $N \ni m > m_0'$ implies

$$\|u_2(x) - u_2^m(x)\| + \|u_1(x) - u_1^m(x)\| < \frac{\varepsilon}{2}. \tag{4.23}$$

Furthermore, due to the present assumptions and Theorem in [8], we can assert that the functions u_μ^m $(\mu = 1,2)$ are of the form (3.1), where R_p^m and $\psi_{i,p-j}^m$ are as on p. 12.

Basing on (1.3) and using an argument analogous to that applied in the proof of (4.20), we get

$$\|u_2^m(x) - u_1^m(x)\| \leq \frac{\varepsilon}{2} \tag{4.24}$$

for $N \ni m > n_0''$, n_0'' $(n_0'' = n_0''(\varepsilon))$ being a sufficiently large positive integer.

On joining (4.22)–(4.24) we can conclude that $u_1 = u_2$ in Ω. Q.E.D.

Finally, we have the following theorem

THEOREM 4.3. *If Assumptions* I–III *of the present paper are replaced by those in paper* [8], *then the g.s. of the* $(\mathfrak{G})$*-problem given by Theorem* 4.1 *is a c.s of this problem.*

The validity of this theorem follows from the results obtained in [8].

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ON STOCHASTIC DIFFERENCE EQUATIONS ASSOCIATED WITH QUASI-DIFFUSION PROCESSES

We consider a sequence of stochastic difference equations. We define the notion "the consistency" of this sequence. This consistent sequence is an analogue of consistent sequence of solutions of Kołmogorow type parabolic equations. We give relations between coefficients of these parabolic equations and coefficients of equations considered in the present paper. We find solutions when the coefficients are linear. Every difference equation of the considered sequence has a form similar as in ARiMA models.

1. Introduction and formulation of the results

Let $(\Omega, \mathfrak{F}, P)$ be a complete probability space, $\mathfrak{F} = (\mathfrak{F}_t,\ t \in [0,T])$ an increasing family of sub σ-field of $\mathfrak{F}$, $W = (W_t, \mathfrak{F}_t)$ a Wiener process.

Let $X = (X_t)$, $t \in [0,T]$, be a real-valued continuous stochastic process $(\mathfrak{F}_t)$-adapted such that $E(X_t) = 0$, $E(X_t^2) < \infty$.

For $0 \le t_1 \le t_2 \le \ldots \le T$ we put $\mathbf{t}_n = (t_1, t_2, \ldots, t_n)$, $\mathbf{X}_n = (X_{t_1}, \ldots, X_{t_n})$ and $\mathbf{x}_n = (x_1, \ldots, x_n) \in R^n$. We suppose that for every n, every $\mathbf{t}_n$ the random variables $X_{t_1}, \ldots, X_{t_n}$ are linearly independent.

Let $a_n(\mathbf{t}_{n+1}, \mathbf{x}_n)$ and $b_n(\mathbf{t}_{n+1}, \mathbf{x}_n)$ be continuous functions.

For a given probability space $(\Omega, \mathfrak{F}, P)$ and a given Wiener process W, we consider a stochastic process X satisfying for every n, every fixed sequence $t_1 < t_2 < \ldots < t_n$ and optional t_{n+1} stochastic difference equations

$$(1.1)\quad X_{t_{n+1}} = X_{t_n} + \int_{t_n}^{t_{n+1}} a_n(\mathbf{t}_n, s, \mathbf{X}_n)\,ds + \int_{t_n}^{t_{n+1}} b_n(\mathbf{t}_n, s, \mathbf{X}_n)\,dW_s = F_n(\mathbf{t}_{n+1}, \mathbf{X}_n, W), \quad n \ge 1.$$

We shall say that the solutions of system (1.1) are a consistent family (CF)

of solutions if for every n the random functionals F_n satisfy the relations

$$(1.2)\quad F_n(\mathbf{t}_{n+1}, \mathbf{X}_{n-1}, F_{n-1}(\mathbf{t}_n, \mathbf{X}_{n-1}, W), W) \overset{\mathrm{d}}{=} \\ \overset{\mathrm{d}}{=} F_{n-1}(\mathbf{t}_{n-1}, t_{n+1}, \mathbf{X}_{n-1}, W).$$

By properties of Ito integrals, almost surely,

$$(1.3)\quad \begin{cases} E(X_{t_{n+1}} \mid \mathfrak{F}_{t_n}) = X_{t_n} + \int\limits_{t_n}^{t_{n+1}} a_n(\mathbf{t}_n, s, \mathbf{X}_n)\, ds, \\ E(X_{t_{n+1}} - E(X_{t_{n+1}} \mid \mathfrak{F}_{t_n}))^2 \mid \mathfrak{F}_{t_n}) = \int\limits_{t_n}^{t_{n+1}} b_n^2(\mathbf{t}_{n+1}, s, \mathbf{X}_n)\, ds. \end{cases}$$

If X is a Markov process the functions a_n, b_n of $2n+1$ arguments turn into the functions of 3 arguments

$$a_n(\mathbf{t}_n, s, \mathbf{X}_n) = a_1(t_n, s, X_{t_n}), \quad b_n(\mathbf{t}_n, s, \mathbf{X}_n) = b_1(t_n, s, X_{t_n}).$$

Therefore the consistency conditions (1.2) reduce to the one condition:

$$(1.2')\qquad F_1(t_2, t_3, F_1(t_1, t_2, X_{t_1}, W), W) \overset{\mathrm{d}}{=} F_1(t_1, t_3, X_{t_1}, W).$$

Formula (1.2′) is similar to a property of semigroups operators for Markov processes. In a non-markovian case for the considered process X we have the system of stochastic difference equations (1.1) satisfying (1.2).

The main aim of this paper is to prove the following Proposition.

PROPOSITION 1. *If $a_n(\mathbf{t}_{n+1}, \mathbf{x}_n) \in C^0$ are linear functions of $\mathbf{x}_n$, $b_n(\mathbf{t}_{n+1}, \mathbf{x}_n) = b_n(\mathbf{t}_{n+1})$ are continuous positive functions, then solutions of (1.1) are (CF) gaussian solutions. For every $n > 1$*

$$(1.4)\quad \operatorname{Var}\Big(X_{t_{n+1}} - X_{t_n} - \int\limits_{t_n}^{t_{n+1}} a_n(\mathbf{t}_n, s, \mathbf{X}_n)\, ds\Big) \le \\ \le \operatorname{Var}\Big(X_{t_{n+1}} - X_{t_n} - \int\limits_{t_n}^{t_{n+1}} a_{n-1}(t_2, \dots, t_n, s, X_{t_2}, \dots, X_{t_n})\, ds\Big).$$

For fixed n equation (1.1) permit to express the state $X_{t_{n+1}}$ at the point t_{n+1} by W and preceding states of the considered process, i.e. by the states at the points $t_1, \dots, t_n$. In this sense we get forecasting at the point t_{n+1} depending on W and the preceding states.

By virtue of (1.4) we can say that such forecasting given by (1.1) for n past states is better than for $n-1$ past states. The greater number of conditions improve the forecasting.

The idea of the sequence of stochastic difference equations (1.1) satisfying (1.2) refers to various stochastic investigations. We mention some relations.

De Haan and Karandicar [2] have considered stochastic difference equation of the form

$$X_t = A_t^s X_s + B_t^s$$

with the random functionals satisfying almost surely

$$\text{(i)} \qquad A_t^s = A_u^s A_t^u, \qquad B_t^s = A_t^u B_u^s + B_t^u \quad \text{for } 0 \leq s \leq u \leq t$$

and some further conditions (ii), (iii). For fixed n relation (1.1) is a stochastic difference equation with memory depending on the preceding states.

Relations (1.2) have similar character to (i). Relations (1.2) for linear coefficients a_n, b_n have a strict connection with (i). The essential difference consists on the dependence of functionals F_n on the precedings states.

On the other hand by (1.3) equations (1.1) can be written in the following form

$$(1.5) \qquad X_{t_{n+1}} - E(X_{t_{n+1}} \mid \mathfrak{F}_{t_n}) = \int\limits_{t_n}^{t_{n+1}} b_n(\mathbf{t}_n, s, \mathbf{X}_n)\, dW_s, \qquad n \geq 1.$$

Every equation of system (1.5) can be treated as a version of Clark's formula with a special form of the functional under the integral sign. Clark's formula was considered for example by Karatzas, Ocone and Jinlu Li [3].

It seems also interesting to mention that under the assumptions of Proposition 1 relations (1.1) are a continuous analogue of ARIMA models (see e.g. Box, Jenkins [1], Priestley [7]).

The most essential connection of the sequence (1.1) is with the quasi-diffusion processes (the definition is quoted in pargraph 2)

The main idea of quasi-diffusion processes is to give a tool for finding conditional distributions. This tool is a sequence of Kolmogorov type parabolic equations. If we have the initial condition and we solve first n equations of this sequence we get n-dimensionals distributions. For a Markov process the solution of Kolmogorov equation determines all multi-dimensionsl distributions. For non-markovian process the n-dimesional distributions (for finite n) give only some partial information; when n increases we get more informations. Now we propose a method to express the state of the process at the moment t_{n+1} by the preceding states of the process at the moments $t_1, t_2, \ldots, t_n$ plus some functional of the Wiener process. This method is based on stochastic difference equations (1.1). The coefficients of equations (1.1) have the strict connection with the coefficients of Kolmogorov type parabolic equation (see chapter 2). When n increases we get better information in the sense of relation (1.4).

Therefore we extend the idea of quqsi-diffusion processes permitting to find conditional distributons (conditioning by preceding states) to the idea of expressing the state of the process by some preceding states plus some functionals. The idea of quasi-diffusion processes is based on infinitesimal moments (some limits of conditional moments). The idea of the present paper is based on conditional moments.

2. Quasi-diffusion processes

Solutions of (1.1) have a strict connection with quasi-diffusion processes. For quasi-diffusion processes conconsidered by Plucińska [4], [5] there exist limits

$$\lim_{\Delta_n \to 0+} \frac{1}{\Delta_n} E(X_{t_n+\Delta_n} - X_{t_n}) \mid \mathbf{X}_n = \mathbf{x}_n) = \widehat{a}_n(\mathbf{t}_n, \mathbf{x}_n), \tag{2.1}$$

$$\lim_{\Delta_n \to 0+} \frac{1}{\Delta_n} E(X_{t_{n+\Delta_n}} - X_{t_n})^2 \mid \mathbf{X}_n = \mathbf{x}_n) = \widehat{b}_n(\mathbf{t}_n, \mathbf{x}_n). \tag{2.2}$$

The conditional densities f_n of quasi-diffusion processes satisfy (under some additional regularity assumptions) the Kolmogorov type equations

$$\begin{aligned} \frac{\partial}{\partial t_n} f_n(\mathbf{t}_{n-1}, \mathbf{x}_{n-1}; t_n, x_n) + \frac{\partial}{\partial x_n}[\widehat{a}_n(\mathbf{t}_n, \mathbf{x}_n) f_n(\mathbf{t}_{n-1}, \mathbf{x}_{n-1}; t_n, x_n)] = \\ = \frac{1}{2}\frac{\partial^2}{\partial x_n^2}[\widehat{b}_n(\mathbf{t}_n, \mathbf{x}_n) f_n(\mathbf{t}_{n-1}, \mathbf{x}_{n-1}; t_n, x_n)], \qquad n > 1. \end{aligned} \tag{2.3}$$

It is obvious that for solutions of (1.1) there exist limits (2.1) and (2.2) and the following relations hold:

$$\begin{cases} \widehat{a}_n(\mathbf{t}_n, \mathbf{x}_n) = a_n(\mathbf{t}_n, t_n, \mathbf{x}_n) \\ \widehat{b}_n(\mathbf{t}_n, \mathbf{x}_n) = b_n^2(\mathbf{t}_n, t_n, \mathbf{x}_n). \end{cases} \tag{2.4}$$

For Markov processes, for $n = 2$, equation (2.3) with some initial conditions determines all finite dimensional distributions. In the non-markovian case we have the sequence of equations (2.3). These equations give a partial information about multi-dimensional distributions. If we consider these equations for $n \leq N$, then we can find the N-dimensional distributions.

Similarly, if we consider equations (1.1) for $n \leq N$ with given a_n, b_n, then by (2.4) and (2.3) we can find conditional densities f_n and next N-dimensional densities.

On the other hand, for given n, equation (1.1) provides a forecasting for $X_{t_{n+1}}$ as a functional of n past states $X_{t_1}, \ldots, X_{t_n}$ and a Wiener process W. A forecasting given by (1.1) for n past states is better than for $n-1$ past states in the sense of (1.4).

3. Auxiliary results

We shall use the following lemmas.

LEMMA 1. *Let $X = (X_t)$, $t \in [0,T]$ be a zero mean stochastic process with $E(X_t^4) < \infty$ and with a continuous covariance function $k_{ij} = E(X_{t_i} X_{t_j})$. If for every n, $t_1 < t_2 < \ldots < T$, the random variables*

(3.1) $X_{t_1}, X_{t_2}, \ldots, X_{t_n}$ *are linearly independent,*

(3.2) $\mu_n = E(X_{t_n} \mid X_{t_1}, \ldots, X_{t_{n-1}})$ *is a linear function of* $X_{t_1}, \ldots, X_{t_{n-1}}$,

(3.3) $v_n^2 = \mathrm{Var}(X_{t_n} \mid X_{t_1}, \ldots, X_{t_{n-1}}) = E[(X_{t_n} - \mu_n)^2 \mid X_{t_1}, \ldots, X_{t_{n-1}}]$ *is a deterministic function*

(3.4) $$E[(X_{t_n} - \mu_n)^4 \mid X_{t_1}, \ldots, X_{t_{n-1}}] \le o(t_n - t_{n-1})$$

then X is a gaussian process. Moreover

(3.5) $$\mu_n = \sum_{i=1}^{n-1} c_{in}(\mathbf{t}_n) X_{t_i},$$

(3.6) $$v_n^2 = \frac{K^{(n)}}{K_{nn}^{(n)}},$$

where

(3.7) $$c_{in} = -\frac{K_{in}^{(n)}}{K^{(n-1)}},$$

$k_{ij} = E(X_{t_i} X_{t_j})$, $K_{in}^{(n)}$ *is the cofactor of k_{in} in the matrix*

$$[k_{ij}]_{i,j=1}^{n},\ K^{(n-1)} = \det[k_{i,j}]_{i,j=1}^{n-1} \quad (i.e.\ K^{(n-1)} = K_{nn}^{(n)}).$$

Evidently for every i the coefficients c_{in} are functions of n arguments: $t_1, \ldots, t_n$.

We omit the proof of Lemma 1 because it is analogous to the proof of Theorem 1 given by Plucińska [6]. In that paper the conditions of type (3.2), (3.3) are assumed for all the point $(t_1, \ldots, t_n)$ (not necesserity ordered). In the present paper conditions (3.2), (3.3) must be satisfied only for $t_1 < t_2 < \ldots < t_n$. But in the present paper we have the additional condition (3.4).

LEMMA 2. *If conditions (3.1), (3.2) and (3.3) hold then*

(3.8) $$c_{i,n+1}(\mathbf{t}_{n+1}) + c_{n,n+1}(\mathbf{t}_{n+1}) c_{i,n}(\mathbf{t}_n) = c_{i,n}(\mathbf{t}_{n-1}, t_{n+1}),$$

(3.9) $$c_{n,n+1}^2(\mathbf{t}_{n+1}) \mathrm{Var}(X_{t_n} \mid X_{t_1}, \ldots, X_{t_{n-1}}) + \mathrm{Var}(X_{t_{n+1}} \mid X_{t_1}, \ldots, X_{t_n}) = \mathrm{Var}(X_{t_{n+1}} \mid X_{t_1}, \ldots, X_{t_{n-1}}).$$

Proof of Lemma 2. We shall use the following formula for determinants of symmetric matrices

$$K^{(n+1)}K_{ij}^{(n)} = K_{ij}^{(n+1)}K_{n+1,n+1}^{(n+1)} - K_{i,n+1}^{(n+1)}K_{j,n+1}^{(n+1)}. \tag{3.10}$$

By virtue of (3.7) and (3.10) we have

$$K^{(n+1)}c_{i,n+1}(\mathbf{t}_{n+1}) = -\frac{K_{i,n+1}^{(n+1)}}{K^{(n)}K^{(n-1)}}[K_{nn}^{(n+1)}K_{n+1,n+1}^{(n+1)} - (K_{n,n+1}^{(n+1)})^2], \tag{3.11}$$

$$c_{n,n+1}(\mathbf{t}_{n+1})c_{i,n}(\mathbf{t}_n) = \frac{K_{n,n+1}^{(n+1)}}{K^{(n)}}\frac{K_{i,n,n+1,n+1}^{(n+1)}}{K_{n,n,n+1,n+1}^{(n+1)}} = \frac{K_{n,n+1}^{(n+1)}}{K^{(n-1)}K^{(n)}K^{(n+1)}}[K_{in}^{(n+1)}K_{n+1,n+1}^{(n+1)} - K_{i,n+1}^{(n+1)}K_{n,n+1}^{(n+1)}], \tag{3.12}$$

$$c_{in}(\mathbf{t}_{n-1}, t_{n+1}) = -\frac{K_{i,n,n,n+1}^{(n+1)}}{K_{n,n,n+1,n+1}^{(n+1)}} = \frac{-K_{i,n+1}^{(n+1)}K_{nn}^{(n+1)} + K_{in}^{(n+1)}K_{n,n+1}^{(n+1)}}{K^{(n-1)}K^{(n+1)}}. \tag{3.13}$$

Formula (3.8) follows immediately from (3.11), (3.12) and (3.13).

Taking into account (3.6), (3.7) and (3.8) we have

$$c_{n,n+1}^2(\mathbf{t}_{n+1})\operatorname{Var}(X_{t_n} \mid X_{t_1}, \ldots, X_{t_{n-1}}) = \left[\frac{K_{n,n+1}^{(n+1)}}{K^{(n+1)}}\right]^2 \frac{K^{(n)}}{K^{(n-1)}} = \frac{[K_{n,n+1}^{(n+1)}]^2}{K^{(n)}K^{(n-1)}}, \tag{3.14}$$

$$\operatorname{Var}(X_{t_{n+1}} \mid X_{t_1}, \ldots, X_{t_n}) = \frac{K^{(n+1)}}{K^{(n)}} = \frac{K^{(n+1)}[K_{nn}^{(n+1)}K_{n+1,n+1}^{(n+1)} - (K_{n,n+1}^{(n+1)})^2]}{K^{(n)}K^{(n-1)}K^{(n+1)}}, \tag{3.15}$$

$$\operatorname{Var}(X_{t_{n+1}} \mid X_{t_1}, \ldots, X_{t_{n-1}}) = \frac{K_{nn}^{(n+1)}}{K^{(n-1)}}. \tag{3.16}$$

Formula (3.9) follows immediately from (3.14), (3.15) and (3.16). Thus Lemma 2 is proved.

4. Proof of Proposition 1

First we are going to show that conditions (1.2) hold. It follows from the

properties of the Itô integral and the linearity of a_n that

$$(4.1)\quad E(X_{t_{n+1}} \mid \mathfrak{F}_{t_n}) = X_{t_n} + \int_{t_n}^{t_{n+1}} a_n(\mathbf{t}_n, s, \mathbf{X}_n)\, ds = \sum_{i=1}^{n} \alpha_{i,n+1} X_{t_i},$$

in other words the conditional expectation is a linear function of the states with some coefficients $\alpha_{i,n+1}$. Then by Lemma 1

$$(4.2)\quad \alpha_{i,n+1} = c_{i,n+1}.$$

By the properties of functions b_n and Itô integrals we have

$$(4.3)\quad E\{[X_{t_{n+1}} - E(X_{t_{n+1}} \mid \mathfrak{F}_{t_n})]^2 \mid \mathfrak{F}_{t_n}\} = \int_{t_n}^{t_{n+1}} b_n^2(\mathbf{t}_n, s)\, ds =$$

$$= E[X_{t_{n+1}} - E(X_{t_{n+1}} \mid \mathfrak{F}_{t_n})]^2 > 0.$$

It follows from (4.1), (4.3) that $X_{t_1}, \ldots, X_{t_{n+1}}$ are linearly independent i.e.

$$(4.4)\quad K^{(n+1)} > 0, \quad n = 1, 2, \ldots.$$

Therefore assumptions (3.1), (3.2) and (3.3) are satisfied and in virtue of Lemma 1 we have

$$(4.5)\quad c_{i,n+1}(\mathbf{t}_{n+1}) = -\frac{K_{i,n+1}^{(n+1)}}{K^{(n)}},$$

$$(4.6)\quad b_n^2(\mathbf{t}_{n+1}) = \frac{\partial}{\partial t_{n+1}} v_{n+1}^2 = \frac{\partial}{\partial t_{n+1}} \frac{K^{(n+1)}}{K^{(n)}}.$$

By virtue of formula (3.8) and (4.5)

$$F_n[\mathbf{t}_{n+1}, \mathbf{X}_{n-1}, F_{n-1}(\mathbf{t}_n, \mathbf{X}_{n-1}, W)W] = \sum_{i=1}^{n-1} c_{i,n+1}(\mathbf{t}_{n+1})\mathbf{X}_{t_i} +$$

$$+ c_{n,n+1}(\mathbf{t}_{n+1}) \Big[\sum_{i=1}^{n-1} c_{i,n}(\mathbf{t}_n) X_{t_i} + \int_{t_{n-1}}^{t_n} b_{n-1}(\mathbf{t}_{n-1}, s)\, dW_s \Big] +$$

$$+ \int_{t_n}^{t_{n+1}} b_n(\mathbf{t}_n, s)\, dW_s = \sum_{i=1}^{n-1} c_{in}(\mathbf{t}_{n-1}, t_{n+1}) X_{t_i} +$$

$$+ c_{n,n+1}(\mathbf{t}_{n+1}) \int_{t_{n-1}}^{t_n} b_{n-1}(\mathbf{t}_{n-1})\, dW_s + \int_{t_n}^{t_{n+1}} b_n(\mathbf{t}_n, s)\, dW_s = I_1 + I_2$$

For fixed $t_1, \ldots, t_{n+1}$ the sum

$$I_2 = c_{n,n+1}(\mathbf{t}_{n+1}) \int_{t_{n-1}}^{t_n} b_{n-1}(\mathbf{t}_{n-1}, s)\, dW_s + \int_{t_n}^{t_{n+1}} b_n(\mathbf{t}_n, s)\, dW_s$$

is the sum of two independent gaussian random variables. Then this sum has a gaussian distribution with the mean value equal to zero and by formula (3.9) the variance is equal to

$$E(I_2^2) = c_{n,n+1}^2(\mathbf{t}_{n+1}) \int\limits_{t_{n-1}}^{t_n} b_{n-1}^2(\mathbf{t}_{n-1}, s)\, ds + \int\limits_{t_n}^{t_{n+1}} b_n^2(\mathbf{t}_n, s)\, ds = \tag{4.7}$$

$$= c_{n,n+1}^2(\mathbf{t}_{n+1}) \operatorname{Var}(X_{t_n} \mid X_{t_1}, \ldots, X_{t_{n-1}}) +$$

$$+ \operatorname{Var}(X_{t_{n+1}} \mid X_{t_1}, \ldots, X_{t_n}) = \operatorname{Var}(X_{t_{n+1}} \mid X_{t_1}, \ldots, X_{t_{n-1}}) =$$

$$= \int\limits_{t_{n-1}}^{t_n} b_{n-1}^2(\mathbf{t}_{n-1}, s)\, ds = E\Big(\int\limits_{t_{n-1}}^{t_{n+1}} b_{n-1}(\mathbf{t}_{n-1}, s)\, dW_s \Big)^2.$$

On the other hand

$$F_{n-1}(\mathbf{t}_{n-1}, t_{n+1}, \mathbf{X}_{n-1}, W) = \sum_{i=1}^{n-1} c_{in}(\mathbf{t}_{n-1}, t_{n+1}) X_{t_i} +$$

$$+ \int\limits_{t_{n-1}}^{t_{n+1}} b_{n-1}(\mathbf{t}_{n-1}, s)\, dW_s = I_1 + I_3.$$

It is evident that for fixed $t_1, \ldots, t_{n+1}$ the integral I_3 has a mean zero gaussian distribution and by (4.7) $I_2 \overset{\mathrm{d}}{=} I_3$. It follows from the indepedence of I_1, I_2 and the indepedence of I_1, I_3 that $I_1 + I_2 \overset{\mathrm{d}}{=} I_1 + I_3$. Thus (1.2) is proved.

Now we are going to show, by Lemma 1, that the solutions of (1.1) are gaussian. Conditions (3.1)–(3.3) follows from (4.1), (4.3) and (4.4). For $\delta = 2$ by virtue of properties of stochastic integrals and the continuity of functions b_n we have

$$E\{(X_{t_{n+1}} - E(X_{t_{n+1}} \mid \mathfrak{F}_{t_{n+1}}))^4 \mid \mathfrak{F}_{t_n}\} = E\Big(\int\limits_{t_n}^{t_{n+1}} b_n(\mathbf{t}_n, s)\, d_s \Big)^4 \leq$$

$$\leq 36(t_{n+1} - t_n) \int\limits_{t_n}^{t_{n+1}} b_n^4(\mathbf{t}_n s)\, ds = o(t_{n+1} - t_n).$$

Thus (3.4) holds. Therefore by virtue of Lemma 1 solutions of (1.1) are gaussian.

Now we are going to show (1.4). Taking into account (3.10), for $i = j = 1$, we have

$$K^{(n+1)} K_{11}^{(n)} = K_{11}^{(n+1)} K^{(n)} - (K_{1,n+1}^{(n+1)})^2. \tag{4.8}$$

It follows from (3.6) and (4.8) that

$$\operatorname{Var}\left(X_{t_{n+1}} - X_{t_n} - \int_{t_n}^{t_{n+1}} a_n(\mathbf{t}_n, s, \mathbf{x}_n)\, ds\right) = \frac{K^{(n+1)}}{K^{(n)}} =$$

$$= \frac{K_{11}^{(n+1)}}{K_{11}^{(n)}} - \frac{(K_{1,n+1}^{(n+1)})^2}{K_{11}^{(n)} K^{(n)}} \leq \frac{K_{11}^{(n+1)}}{K_{11}^{(n)}} =$$

$$= \operatorname{Var}\left(X_{t_{n+1}} - X_{t_n} - \int_{t_n}^{t_{n+1}} a_{n-1}(t_2, \ldots, t_n, s, X_{t_2}, \ldots, X_{t_n})\, ds\right).$$

Formula (1.4) is thus shown. Therefore Proposition 1 is proved.

5. Examples

EXAMPLE 1. Let assumptions of Proposition 1 be satisfied and $k(t_1, t_2) = E(X_{t_1} X_{t_2}) = \exp[-(t_2 - t_1)^2]$. Then the coefficients are given by formulas

$$c_{12}(t_1, t_2) = \exp[-(t_2 - t_1)^2],$$

$$c_{13}(\mathbf{t}_3) = \frac{\exp[-(t_3 - t_2)^2] - \exp[-(t_2 - t_1)^2 - (t_3 - t_2)^2]}{1 - \exp[-2(t_2 - t_1)^2]}$$

$$c_{23}(\mathbf{t}_3) = \frac{\exp[-(t_3 - t_2)^2] - \exp[-(t_2 - t_1)^2 - (t_3 - t_1)^2]}{1 - \exp[-2(t_2 - t_1)^2]}$$

$$b_1^2(\mathbf{t}_2) = \frac{\partial}{\partial t_2}\{1 - \exp[-2(t_2 - t_1)^2]\}$$

$$b_2^2(\mathbf{t}_3) = \frac{\partial}{\partial t_3}\{(1 + 2\exp[-(t_2 - t_1)^2 - (t_3 - t_2)^2 - (t_3 - t_1)^2] -$$
$$- \exp[-2(t_3 - t_1)] - \exp[-2(t_3 - t_2)^2] - \exp[-2(t_2 - t_1)^2]) \times$$
$$\times (1 - \exp[-2(t_2 - t_1)^2])^{-1}\}.$$

Formula (1.2) for $n = 3$ has, in virtue of (1.5), the following form

$$X_{t_3} = F_2(t_1, t_2, t_3, X_{t_1}, F_1(t_1, t_2, W), W) =$$

$$= c_{13}(\mathbf{t}_3) X_{t_1} + c_{23}(\mathbf{t}_3)\left[c_{12}(\mathbf{t}_2) X_{t_1} + \int_{t_1}^{t_2} b_1(t_1, s)\, dW_s\right] +$$

$$+ \int_{t_2}^{t_3} b_2(t_1, t_2, s)\, dW_s \stackrel{\mathrm{d}}{=} c_{12}(t_1, t_3) X_{t_1} + \int_{t_1}^{t_3} b_1(t, s)\, dW_s =$$

$$= F_1(t_1, t_3, W).$$

Evidently we have

$$\frac{K^{(3)}}{K^{(2)}} \leq \frac{K_{11}^{(3)}}{K_{11}^{(2)}}.$$

Therefore formula (1.4) is satisfied.

EXAMPLE 2. An example of a stochastic process satisfying (1.1) is Ornstein–Uhlenbeck [8] process

$$X_t = e^{-\rho t} X_0 + \int_0^t e^{-\rho(t-u)}\, dW_u.$$

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ON THE SOLUTION OF THE FIRST FOURIER PROBLEM FOR THE SYSTEM OF DIFFUSION EQUATIONS

1. Introduction

In this paper we are studying the first Fourier problem (F) for the parabolic (in Petrovskii's sense) system

$$v_t(t,x) = \mathbb{A}(\Delta v)(t,x) + \varphi(t,x) \qquad (t,x) \in \,]0,T[\, \times \Omega,$$

with $\mathbb{A}$ given real $k \times k$ matrix, $\varphi : [0,T] \times \overline{\Omega} \ni (t,x) \to \varphi(t,x) \in \mathbb{R}^k$ given function and $v : [0,T] \times \overline{\Omega} \ni (t,x) \to v(t,x) \in \mathbb{R}^k$, unknown function where $\Omega = \{(x_1, x_2) : x_1^2 + x_2^2 \le a^2\}$.

The initial condition

$$v(0,x) = g(x), \quad x \in \overline{\Omega}, \tag{1.2}$$

where $g : \overline{\Omega} \ni x \to g(x) \in \mathbb{R}^k$ is a given function and boundary condition

$$v(t,x) = 0 \quad \text{on } [0,T] \times \partial\Omega \tag{1.3}$$

are considered. This corresponds physically to diffusion of several gases and evolution of their concentrations.

The solution of this problem is represented as a sum of two integrals being counterparts of the Poisson–Weierstrass integral and potential of plane domain. Kerneks of these integrals are represented by the matrix-function G introduced in this paper and playing crucial role in a representation of a solution v of given Fourier problem (F).

Similar problems were solved by Majchrowski [3] and by Majchrowski and Rogulski [2], but for $\Omega = [0,1]$ only.

2. Assumptions

We make the following assumptions for the functions φ, g and for the matrix $\mathbf{A}$:

(2.1) $\varphi(t,x) = 0$ for $(t,x) \in [0,T] \times \partial\Omega$ and the function $\widetilde{\varphi} : [0,T] \times [0,a] \times [0,2\pi] \to \mathbb{R}^k$, defined by $\widetilde{\varphi}(t,\rho,\gamma) := \varphi(t,\rho\cos\gamma,\rho\sin\gamma)$ is of the class C^1, of the class C^2 in ρ, of the class C^4 in γ, the derivatives $\frac{\partial^{2+l}}{\partial\gamma^l\partial\rho^2}\widetilde{\varphi}$ exist, are continuous, bounded and vanish on $\partial\Omega$ for $l \in \{1,2\}$, and besides $\frac{\partial^s}{\partial\gamma^s}\widetilde{\varphi}(t,\rho,0) = \frac{\partial^s}{\partial\gamma^s}\widetilde{\varphi}(t,\rho,2\pi)$ for $s \in \{0,1,2,3\}$,

(2.2) $g(x) = 0$ for $x \in \partial\Omega$; the function $\widetilde{g} : [0,a] \times [0,2\pi] \to \mathbb{R}^k$, defined by $\widetilde{g}(\rho,\gamma) := g(\rho\cos\gamma,\rho\sin\gamma)$ is of the class C^2 and of the class C^4 in γ, the derivatives $\frac{\partial^{2+l}}{\partial\gamma^l\partial\rho^2}\widetilde{g}$ exist, are continuous, bounded and vanish on $\partial\Omega$ for $l \in \{1,2\}$, and besides $\frac{\partial^s}{\partial\gamma^s}\widetilde{g}(\rho,0) = \frac{\partial^s}{\partial\gamma^s}\widetilde{g}(\rho,2\pi)$ for $s \in \{0,1,2,3\}$,

(2.3) all eigenvalues of $\mathbb{A}$ have positive real parts.

3. Matrix-function $G(t,r,\rho,\vartheta)$

We denote by $\mu_{n,m}$ the m-th positive real zero of equation $J_n(z) = 0$, where $n \in \{0\} \cup \mathbb{N}$, $m \in \mathbb{N}$, and J_n is the Bessel function of degree n.

LEMMA 3.1. *Under the assumption* (2.3) *for every* $T \in]0,\infty[$ *there exist positive constants* C, β, k_0, E *such that for every* $m \in \mathbb{N} \cap [E,\infty]$ *there exists* $l \in \mathbb{N}$ *such that for every* $n \in \mathbb{N}$ *and for every* $t \in]0,T[$

$$\left\| \exp\left[-(\mu_{n,m})^2 \frac{t}{a^2}\mathbb{A}\right]\right\| \le C\exp\left[-(n+l)^2\frac{\pi^2}{4}t\beta\right](n+1)^{2k_0}.$$

Proof. In virtue of [2] (p. 1077), there exists a canonical decomposition $\frac{1}{a^2}\mathbb{A} = S + N$ such that $SN = NS$, where S is a semisimple matrix and $N^{k_0+1} = 0$ for $k_0 \le k-1$. Let $S = BCB^{-1}$, where C is the matrix in the Jordana form for the matrix S, whereas B is the matrix of likeness. Then

$$\begin{aligned}\left\| \exp\left[-(\mu_{n,m})^2 \frac{t}{a^2}\mathbb{A}\right]\right\| &\le \| \exp[-(\mu_{n,m})^2 tS]\exp[-(\mu_{n,m})^2 tN]\| \le \\ &\le \| \exp[-(\mu_{n,m})^2 tS]\|\,\| \exp[-(\mu_{n,m})^2 tN]\| = \\ &= \| B\exp[-(\mu_{n,m})^2 tC]B^{-1}\|\,\| \exp[-(\mu_{n,m})^2 tN]\| \le \\ &\le \|B\|\cdot\|B^{-1}\|\exp[-(\mu_{n,m})^2 t\beta]\|\,\exp[-(\mu_{n,m})^2 tN]\|,\end{aligned}$$

where $\beta = \min(\operatorname{Re}\lambda, \lambda$ is an eigenvalue of the matrix $\frac{1}{a^2}\mathbb{A})$. If we take into consideration the inequality

$$(3.1) \qquad \|\exp[-(\mu_{n,m})^2 tN]\| = \left\| \sum_{j=0}^{k_0} \frac{[-(\mu_{n,m})^2 t]^j}{j!} N^j \right\| \leq$$

$$\leq [\max\{1, (\mu_{n,m})^2\}]^{k_0} \sum_{j=0}^{k_0} \frac{T^j}{j!} \|N\|^j,$$

then we have

$$\left\| \exp -(\mu_{n,m})^2 \frac{t}{a^2} \mathbb{A} \right\| \leq$$

$$\leq \|B\| \cdot \|B^{-1}\| \exp[-(\mu_{n,m})^2 t\beta](\mu_{n,m})^{2k_0} \sum_{j=0}^{k_0} \frac{T^j}{j!} \|N\|^j.$$

The last inequality follows from the inequalities $\mu_{n,m} \geq 1$ and $\mu_{n,m} > n$ for each $(n,m) \in \mathbb{N} \times \mathbb{N}$ and from the fact that there exists $\overline{k} \in \mathbb{N} \cup \{0\}$ such that $\mu_{0,m} \in]\overline{k}\pi + \frac{3}{4}\pi, \overline{k}\pi + \frac{7}{8}\pi[$ for each $m \in \mathbb{N}$ (see [4] p. 485 and p. 490). From Hankel's asymptotic formula for $x \to \infty$ (see [4] p. 488)

$$J_n(x) = \sqrt{\frac{2}{\pi x}} \left[\cos\left(x - \frac{n\pi}{2} \right) + O\left(\frac{1}{x} \right) \right]$$

it follows that $\cos(\mu_{n,m} - \frac{n\pi}{2}) = O(\frac{1}{\mu_{n,m}})$. Consequently, there exists a constant $\delta > 0$ such that

$$-\delta + \frac{\pi}{2}(n+1) + l\pi \leq \mu_{n,m} \leq \delta + \frac{\pi}{2}(n+1) + l\pi$$

holds for sufficiently large $m \in \mathbb{N}$ and for some $l \in \mathbb{N}$ because of the inequality $0 < \mu_{n,m} < \mu_{n,m+1}$ for $(n,m) \in (\mathbb{N} \cup \{0\}) \times \mathbb{N}$. Hence, for sufficiently large $m \in \mathbb{N}$, there exists $l \in \mathbb{N}$ such that for each $n \in \mathbb{N} \cup \{0\}$ we have

$$(3.2) \qquad \frac{1}{2}(n+l)\pi \leq \mu_{n,m} \leq (n+l)\pi.$$

Finally

$$\left\| \exp\left[-(\mu_{n,m})^2 \frac{t}{a^2} \mathbb{A} \right] \right\| \leq C \exp\left[-(n+l)^2 \cdot \frac{\pi^2}{4} r\beta \right] (n+l)^{2k_0}.$$

This ends the proof of Lemma 3.1.

LEMMA 3.2. *Under the assumption (2.3) there exists a constant $b \in \mathbb{R}$ such that for each $t > 0$ the inequality $\|\exp[-(\mu_{n,m})^2 \frac{t}{a^2}\mathbb{A}]\| \leq b$ holds.*

Proof. Observe that (see [2])

$$e^{-k^2 z}k^{2j} \le \frac{j^j}{z^j}e^{-j} \quad \text{for all } j,k,z>0.$$

Then from (3.1) we obtain the inequality

$$\begin{aligned}
&\|\exp[-(\mu_{n,m})^2 t(S+N)]\| \le \\
&\le \|B\|\cdot\|B^{-1}\|\sum_{j=0}^{k_0}\exp\Big[-(n+l)^2\frac{\pi^2}{16}t\beta\Big]\frac{(n+l)^{2j}\pi^{2j}t^j}{j!}\|N\|^j \le \\
&\le \|B\|\cdot\|B^{-1}\|\Big(1+\sum_{j=1}^{k_0}\frac{(4j\|N\|)^j}{\beta^j j!}e^{-j}\Big)
\end{aligned}$$

for sufficiently large m, for instance $m > m_0$. The methods applied above allow us also to show for $m \le m_0$ that all the functions of the form

$$\mathbb{R}_+ \ni t \to \Big\|\exp\Big[-\Big(\frac{\mu_{n,m}}{a}\Big)^2 t\mathbb{A}\Big]\Big\| \in \mathbb{R}$$

are bounded. The proof of Lemma 3.2 is complete.

Let us introduce now a matrix-valued function

$$G :]0,\infty[\times [0,a]\times[0,a]\times\mathbb{R} \to \mathbb{R}^{k^2}$$

given by the formula

$$(3.3)\quad G(t,r,\rho,\vartheta) = \\ = \sum_{n=0}^{\infty}\sum_{n=1}^{\infty}\frac{J_n(\mu_{n,m}\frac{r}{a})J_n(\mu_{n,m}\frac{\rho}{a})}{\varepsilon_n[J_{n+1}(\mu_{n,m})]^2}\cos n\vartheta\cdot\exp\Big[-(\mu_{n,m})^2\frac{t\mathbb{A}}{a^2}\Big],$$

with

$$\varepsilon_n = \begin{cases} 2 & \text{for } n=0,\\ 1 & \text{for } n>0,\end{cases}$$

which plays the same role for the system (1.1) as the function θ_3 for a single parabolic equation in Cannon's paper [1] or the matrix-function M introduced by Majchrowski and Rogulski ([2]).

Now we shall consider the properties of the matrix-function G.

THEOREM 3.1. *Under the assumption (2.3) the matrix-valued function G defined by the formula (3.3) is of the class C^∞ and all its partial derivatives can be calculated by term-by-term differentiation of the series (3.3).*

Proof. From the fact that $J_n(z)$ and $I_{n+l}(z)$, where $l \in \mathbb{N}$, have no common roots (see [4] p. 484), from the inequality $|J_n(x)| \leq 1$ for $n \in \mathbb{N} \cup \{0\}$ and $x \in \mathbb{R}$ (see [4] p. 31), and from Hankel's asymptotic formula we obtain the inequality

$$\left| \frac{J_n(\mu_{n,m}\frac{r}{a})J_n(\mu_{n,m}\frac{\rho}{a})}{\varepsilon_n[J_{n+1}(\mu_{n,m})]^2} \right| \leq C$$

for all $n \in \mathbb{N} \cup \{0\}$, $m \in \mathbb{N}$ and $r, \rho \in [0, a]$, where C is a constant independent of r, ρ, m, n. It follows from Lemma 3.1 that the series is uniformly convergent on the arbitrary subset

$$P_\delta = \{(t, r, \rho, \vartheta) : \delta < t < T; r, \rho \in [O, a];\ \vartheta \in \mathbb{R}\}$$

of the set $]0, \infty[\times [0, a] \times [0, a] \times \mathbb{R}$.

Taking also into consideration Lemma 3.2 we complete the proof of theorem.

4. Auxiliary theorems

LEMMA 4.1. *If*

1° $\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} c_{n,m}$ *is convergent,*

2° *the functions* $f_{n,m} : [0, T] \to \mathbb{R}$, *where* $(n, m) \in J = (\mathbb{N} \cup \{0\}) \times \mathbb{N}$, *have the properties*

(a) $\bigwedge_{(n,m)\in J} \bigwedge_{\delta\in]0,T[} \bigvee_{q\in]0,1[} \bigvee_{l(m)\in\mathbb{N}} \bigwedge_{t\in[\delta,T]} (\lim_{m\to\infty} l(m) = \infty \wedge 0 \leq f_{n,m}(t) \leq q^{n+l(m)}),$

(b) $\bigwedge_{(n,m)\in J} \lim_{t\to 0+} f_{n,m}(t) = 1 = f_{n,m}(0),$

(c) $\bigwedge_{t\in[0,T]} \bigwedge_{(n,m)\in J} ((f_{n+1,m}(t) \leq f_{n,m}(t)) \wedge (f_{n,m+1}(t) \leq f_{n,m}(t))),$

then

1) $\bigwedge_{\delta\in[0,T]} \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}(c_{n,m}f_{n,m}(\cdot))$ *is uniformlý convergent on* $[\delta, T]$,

2) *there exists*

$$\lim_{t\to 0+} \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} c_{n,m}f_{n,m}(t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} c_{n,m}.$$

Proof. At first we shall prove that for every $\varepsilon > 0$ there exists $p_0 \in \mathbb{N}$, such that for all $k > p_0$ and $r > p_0$ and all $p, q \in \mathbb{N}$ such that $p \geq k$ and $q \geq r$ the following inequality holds

$$\left|\sum_{n=0}^{p}\sum_{m=1}^{q} c_{n,m} f_{n,m}(t) - \sum_{n=0}^{k-1}\sum_{m=1}^{r-1} c_{n,m} f_{n,m}(t)\right| < \varepsilon$$

which we can rewrite as

$$\left|\sum_{n=0}^{p}\sum_{m=r}^{q} c_{n,m} f_{n,m}(t) + \sum_{n=k}^{p}\sum_{m=1}^{r-1} c_{n,m} f_{n,m}(t)\right| < \varepsilon.$$

Let us denote

$$S_1(t) := \sum_{n=0}^{p}\sum_{m=r}^{q} c_{n,m} f_{n,m}(t), \quad S_2(t) := \sum_{n=k}^{p}\sum_{m=1}^{r-1} c_{n,m} f_{n,m}(t),$$

$$\sigma_w^{(n)} = \sum_{s=1}^{w} c_{n,s}, \quad \Delta_{w,r} := \sum_{s=0}^{w} M_r^{(s)}, \quad \delta_{w,r} := \sum_{s=0}^{w} m_r^{(s)},$$

where

$$M_r^{(s)} := \begin{cases} \max(\sigma_1^{(s)}, \ldots, \sigma_{r-1}^{(s)}, -\sigma_{r-1}^{(s)}, \sigma_r^{(s)}, \ldots, \sigma_q^{(s)}) & \text{for } 1 < r \leq q, \\ \max(\sigma_1^{(s)}, \ldots, \sigma_q^{(s)}) & \text{for } r = 1, \text{ and for } s \in \mathbb{N}, \end{cases}$$

$$m_r^{(s)} := \begin{cases} \min(\sigma_q^{(s)}, \ldots, \sigma_{r-1}^{(s)}, -\sigma_{r-1}^{(s)}, \sigma_r^{(s)}, \ldots, \sigma_q^{(s)}) & \text{for } 1 < r \leq q, \\ \min(\sigma_1^{(s)}, \ldots, \sigma_q^{(s)}) & \text{for } r = 1, \text{ and for } s \in \mathbb{N} \cup \{0\}, \end{cases}$$

$$M_k := \begin{cases} \max(\Delta_{0,r}, \ldots, \Delta_{k-1,r}, -\Delta_{k-1,r}, \Delta_{k,r}, \ldots, \Delta_{p,r}) & \text{for } 0 < k \leq p, \\ \max(\Delta_{0,r}, \ldots, \Delta_{p,r}) & \text{for } k = 0, \end{cases}$$

$$m_k := \begin{cases} \min(\delta_{0,r}, \ldots, \delta_{k-1,r}, -\delta_{k-1,r}, \delta_{k,r}, \ldots, \delta_{p,r}) & \text{for } 0 < k \leq p, \\ \min(\delta_{0,r}, \ldots, \delta_{p,r}) & \text{for } k = 0. \end{cases}$$

Next, using the Abel transformation, we get the inequality

$$\sum_{m=r}^{q} c_{n,m} f_{n,m}(t) = \sum_{s=0}^{q-r} f_{n,r+s}(t)(\sigma_{r+s}^{(n)} - \sigma_{r+s-1}^{(n)}) =$$

$$= -\sigma_{r-1}^{(n)} f_{n,r}(t) + \sum_{s=0}^{q-1-r} \sigma_{r+s}^{(n)}(f_{n,r+s}(t) - f_{n,r+s+1}(t)) +$$

$$+ \sigma_q^{(n)} f_{n,q}(t) \leq 2M_r^{(n)} f_{n,r}(t)$$

and analogously

$$\sum_{m=r}^{q} c_{n,m} f_{n,m}(t) \geq 2m_r^{(n)} f_{n,r}(t)$$

which imply

$$S_1(t) \leq \sum_{n=0}^{p}(2m_r^{(n)} f_{n,r}(t)) = 2\Big\{ \sum_{s=1}^{p}(\Delta_{s,r} - \Delta_{s-1,r}) f_{s,r}(t) + \Delta_{0,r} f_{0,r}(t)\Big\} =$$

$$= 2\Big\{ \sum_{s=1}^{p} \Delta_{s-1,r}(f_{s-1,r}(t) - f_{s,r}(t)) + \Delta_{p,r} f_{p,r}(t)\Big\}.$$

Finally, we have

$$|S_1(t)| \leq 2D_k f_{0,r}(t) \leq 2D_k q_0^{l(r)},$$

where $D_k = \max(|M_k|, |m_k|)$. The last inequality holds independently of $t \in [\delta, T]$ and for all $\delta \in]0, T]$.

Using the Abel transformation, we have

$$\sum_{m=1}^{r-1} c_{n,m} f_{n,m}(t) = \sigma_1^{(n)} f_{n,1}(t) + \sum_{s=2}^{r-1}(\sigma_s^{(n)} - \sigma_{s-1}^{(n)}) f_{n,s}(t) =$$

$$= \sum_{s=1}^{r-2} \sigma_s^{(n)}(f_{n,s}(t) - f_{n,s+1}(t) + \sigma_{r-1}^{(n)} f_{n,r-1}(t)$$

and, by the assumptions, we obtain the inequalities

$$\sum_{m=1}^{r-1} c_{n,m} f_{n,m}(t) \leq M_1^{(n)} f_{n,1}(t) \leq M_r^{(n)} f_{n,1}(t),$$

$$\sum_{m=1}^{r-1} c_{n,m} f_{n,m}(t) \geq m_1^{(n)} f_{n,1}(t) \geq m_r^{(n)} f_{n,1}(t).$$

Next, the Abel transformation gives

$$\sum_{n=k}^{p} M_r^{(n)} f_{n,1}(t) = -\Delta_{k-1,r} f_{k,1}(t) +$$

$$+ \sum_{n=k}^{p-1} \Delta_{n,r}(f_{n,1}(t) - f_{n+1,1}(t)) + \Delta_{p,r} f_{p,1}(t),$$

and from the assumptions we have $2m_k f_{k,1}(t) \leq S_2(t) \leq 2M_k f_{k,1}(t)$ or $|S_2(t)| \leq 2D_k f_{k,1}(t)$ and then

$$|S_1(t) + S_2(t)| \leq 2D_k(q_0^{l(r)} + q_0^k q_0^{l(1)}) \leq \varepsilon$$

for $k > k_0$, $r > r_0$, where k_0 and r_0 ate sufficiently large non-negative integers, $q_0 \in]0,1[$ and there exists $p_0 = \max(k_0, r_0)$.

This ends the proof of part 1). Part 2) follows immediately from the equality

$$\lim_{t\to 0^+} \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} c_{n,m} f_{n,m}(t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} c_{n,m} f_{n,m}(0). \ \blacksquare$$

LEMMA 4.2. *The functions of the form* $f_{n,m}(t) = \exp[-(\mu_{n,m})^2 \alpha t]$, *where* $\alpha > 0$, $t \in [0,T]$, $(n,m) \in J$ *fulfil assumptions of Lemma* 4.1.

Proof. The inequality (a) of Lemma 4.1 follows immediately from (3.2). Continuity of the functions $f_{n,m}$ is evident. From [4] (p. 479) it follows that the positive zeros of $J_n(x)$ are interlaced with those of $J_{n+1}(x)$, i.e. $0 < \ldots < \mu_{n,m} < \mu_{n+1,m} < \mu_{n,m+1} < \mu_{n+1,m+1}$ what implies that for every $t \in [0,T]$ the functions $f_{n,m}$ satisfy 2°(c).

LEMMA 4.3. *If*

1° S *is a semisimple real matrix* $k \times k$ *with eigenvalues* λ *such that* $\operatorname{Re}\lambda > 0$,

2° $\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} c_{n,m}$ *is convergent*,

3° β *is a positive constant*,

then $\bigwedge_{t\geq 0} \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} c_{n,m} \exp(-(\mu_{n,m})^2 \beta t S)$ *is convergent and there exists the limit*

$$\lim_{t\to 0^+} \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} c_{n,m} \exp[-(\mu_{n,m})^2 \beta t S] = \Big(\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} c_{n,m}\Big) I \tag{4.1}$$

where I *is the unit matrix* $k \times k$.

Proof. In view of the relation $\|\exp(-cS)\| \leq D\exp(-c\alpha)$, with $\alpha = \min\{\operatorname{Re}\lambda : \det(S - \lambda I) = 0\}$ and c, D positive constants (see [2]), we have

$$\|\exp[-(\mu_{n,m})^2 \beta t S]\| \leq \|M\|\,\|M^{-1}\|\,\|\exp[-(\mu_{n,m})^2 \alpha\beta t]\|,$$

where M is the matrix of likeness. From the assumptions and from Lemmas 4.1, 4.2 the convergence of considered series follows. From the first part of Lemma 4.3 we obtain (4.1), because the series is uniformly convergent on $[\delta, T]$ for all $\delta \in]0,T[$.

Let us introduce now some denotations which will be used in the next theorem:

$$F : [0, T] \times [0, 1] \times [0, 2\pi] \ni (\eta, \rho, \gamma) \to F(\eta, \rho, \gamma) \in \mathbb{R},$$

$$a_{n,m}(\eta, \gamma) := \frac{2 \int_0^1 \rho F(\eta, \rho, \gamma) J_n(\mu_{n,m}\rho)\, d\rho}{(J_{n+1}(\mu_{n,m}))^2},$$

$$h_{n,m}(\eta, r, \beta) := J_n(\mu_{n,m} r)[\cos n\beta \overset{1}{h}_{n,m}(\eta) + \sin n\beta \overset{2}{h}_{n,m}(\eta)],$$

where

$$\overset{1}{h}_{n,m}(\eta) := \frac{2}{(J_{n+1}(\mu_{n,m}))^2} \int_0^{2\pi} \Big[\int_0^1 \rho \cos n\gamma J_n(\mu_{n,m}\rho) F(\eta, \rho, \gamma)\, d\rho \Big]\, d\gamma,$$

$$\overset{2}{h}_{n,m}(\eta) := \frac{2}{(J_{n+1}(\mu_{n,m}))^2} \int_0^{2\pi} \Big[\int_0^1 \rho \sin n\gamma J_n(\mu_{n,m}\rho) F(\eta, \rho, \gamma)\, d\rho \Big]\, d\gamma,$$

$$c_{n,m}(\eta, \gamma, \beta) := \mu_{n,m}^2 h_{n,m}(\eta, r, \beta),$$

$$(r, \beta) \in [0, 1] \times [0, 2\pi] \quad \text{and} \quad (n, m) \in J.$$

THEOREM 4.1. *If the function F fulfils the same assumptions* (2.1) *as $\tilde{\varphi}$ with $a = 1$ and $k = 1$, then the series $\sum_{n=0}^{\infty} \sum_{m=1}^{\infty} c_{n,m}(\eta, r, \beta)$, $\sum_{n=0}^{\infty} \sum_{m=1}^{\infty} h_{n,m}(\eta, r, \beta)$ are uniformly convergent in η on every interval $[\delta, T - \delta_1]$, where $0 < \delta < T - \delta_1 < T$, for all $(r, \beta) \in [0, 1] \times [0, 2\pi]$.*

Proof. If we denote $a_m := a_{n,m}(\eta, \gamma)$ for fixed η, γ and $n \in \{0\} \cup \mathbb{N}$, then

$$\sum_{m=1}^{\infty} a_m J_n(\mu_{n,m} r)$$

is the Fourier–Bessel series of the function F with respect to ρ. By assumptions, for $l = 0$ and $l = 1$ the derivatives $\frac{\partial^l}{\partial r^l} F(\eta, r, \beta)$ have limited total fluctuation in $[\varepsilon, 1 - \varepsilon]$ for all $\varepsilon \in]0, \frac{1}{2}[$ and there exist the integrals $\int_0^1 \sqrt{\rho} \frac{\partial^l}{\partial \rho^l} F(\eta, \rho, \beta)\, d\rho$, $\int_0^1 \rho^{n+\frac{1}{2}} \frac{\partial}{\partial \rho}(\rho^{-n} \frac{\partial^l}{\partial \rho^l} F(\eta, \rho, \beta))\, d\rho$, $n \in \mathbb{N} \cup \{0\}$, and the limits $\lim_{r \to 0^+} \frac{\partial^l}{\partial r^l} F(\eta, r, \beta) = 0$, $\lim_{r \to 1^-} \frac{\partial^l}{\partial r^l} F(\eta, r, \beta) = 0$. Hence the series $\sum_{m=1}^{\infty} a_m \mu_{n,m} J_n'(\mu_{n,m})$ (see [4] p. 605) is convergent to $\frac{\partial}{\partial r} F(\eta, r, \beta)$ for fixed η, β and n. From the recurrence formulae

$$zJ_n'(z) + nJ_n(z) = zJ_{n-1}(z), \qquad zJ_n'(z) - nJ_n(z) = -zJ_{n+1}(z),$$

for $n \in \{0\} \cup \mathbb{N}$, it follows immediately that

$$
\begin{aligned}
J_n''(z) &= \left(\frac{n}{z}J_n(z) - J_{n+1}(z)\right)' = \\
&= -J_{n+1}'(z) - \frac{n}{z^2}J_n(z) + \frac{n}{z}J_n'(z) = \\
&= \frac{n+1}{z}J_{n+1}(z) - J_n(z) - \frac{n}{z^2}J_n(z) + \frac{n}{z}\left(\frac{n}{z}J_n(z) - J_{n+1}(z)\right) = \\
&= \frac{1}{z}J_{n+1}(z) + \frac{n^2-n}{z^2}J_n(z) - J_n(z).
\end{aligned}
$$

By assumptions on F, the series

$$\sum_{m+1}^{\infty} \frac{n^2 - r}{r^2\mu_{n,m}^2} a_{n,m}^2 J_n(\mu_{n,m} r) = \frac{n^2-n}{r^2}\sum_{m=1}^{\infty} a_m J_n(\mu_{n,m} r)$$

is the Fourier–Bessel expansion of the function F. Analogously, the series $\frac{1}{r}\sum_{m=1}^{\infty} a_m \mu_{n,m} J_{n+1}(\mu_{n,m} r)$ is the Dini expansion of the form $\sum_{m=1}^{\infty} b_m J_{n+1}(\lambda_{n+1,m}\rho)$, where

$$
\begin{aligned}
b_m = 2\lambda_{n+1,m}^2 \int_0^1 t\left(\frac{\partial}{\partial t}F(\eta,t,\beta) - \frac{n}{t}F(\eta,t,\beta)\right) J_{n+1}(\lambda_{n+1,m} t)\, dt \times \\
\times \left\{(\lambda_{n+1,m}^2 - (n+1)^2) J_{n+1}^2(\lambda_{n+1,m}) + \lambda_{n+1,m}^2 (J_{n+1}'(\lambda_{n+1,m}))^2\right\}^{-1},
\end{aligned}
$$

where $\lambda_{n+1,m}$ denotes m-th positive real zero of the function $\{zJ_{n+1}'(z) + (n+1)J_{n+1}(z)\}$. Taking advantage of the recurrence formulae for the Bessel function, we see that $\lambda_{n+1,m} = \mu_{n,m}$ and, in virtue of assumptions, $b_m = a_m\mu_{n,m}$. The assumptions are sufficient to the uniformly convergence of the Fourier–Bessel series and of the Dini series (see [4] p. 593 and p. 601) with respect to the variable r on all intevrals $[\varepsilon, 1-\varepsilon]$, where $0 < \varepsilon < 1$.

Taking into consideration the recurrence formulae since furthermore the series $\sum_{m=1}^{\infty} a_m \mu_{n,m} J_n'(\mu_{n,m} r)$ is convergent to the function $\frac{\partial}{\partial r}F(\eta,r,\beta)$, (see [4] p. 605) and the series $\sum_{m=1}^{\infty} a_m \mu_{n,m}^2 J_n''(\mu_{n,m} r)$ is uniformly convergent to $\frac{\partial^2}{\partial r^2}F(\eta,r,\beta)$, with respect to r, the series $\sum_{m=1}^{\infty} c_{n,m}(\eta,r,\beta)$ is convergent in η for $n \in \{0\} \cup \mathbb{N}$.

Next, applying [4] (p. 583 and p. 598), we can represent the partial sum of a Fourier–Bessel series and of a Dini series as a sum of residues of one function of complex variable having poles at the points $\mu_{n,m}$ in the case of Fourier–Bessel series and at the points $\mu_{n,m}$ and $\lambda_{n,m}$ in the case of Dini series. Therefore, the function which is the sum of the series $\sum_{m=1}^{\infty} c_{n,m}(\eta,r,\beta)$ is continuous with respect to the parameters η and β for all $n \in \{0\} \cup \mathbb{N}$. It follows from the definition of residue and from the

compactness of the contour on which we calculate integrals. This means that the functions whose variables are parameters of integrals are continuous.

Let us denote $S_n(\eta, r, \beta) := \sum_{m=1}^{\infty} c_{n,m}(\eta, r, \beta)$. Making use of the mathematical induction with respect to $p \in \mathbb{N}$ and integrating by parts, one can prove that

$$(4.2) \qquad \int_0^{2\pi} \cos n\gamma F(\eta, \rho, \gamma)\, d\gamma = (-1)^p \frac{1}{n^{2p}} \int_0^{2\pi} \cos n\gamma \frac{\partial^{2p}}{\partial \gamma^{2p}} F(\eta, \rho, \gamma)\, d\gamma,$$

by assumption

$$\frac{\partial^{2s-1}}{\partial \gamma^{2s-1}} G(\eta, \rho, 0) = \frac{\partial^{2s-1}}{\partial \gamma^{2s-1}} F(\eta, \rho, 2\pi)$$

for $s \in \mathbb{N} \cap [1, p]$, $(\eta, \rho) \in [0, T] \times [0, 1]$, and that

$$(4.3) \qquad \int_0^{2\pi} \sin n\gamma F(\eta, \rho, \gamma)\, d\gamma = (-1)^p \frac{1}{n^{2p}} \int_0^{2\pi} \sin n\gamma \frac{\partial^{2p}}{\partial \gamma^{2p}} F(\eta, \rho, \gamma)\, d\gamma,$$

by assumption

$$\frac{\partial^{2(s-1)}}{\partial \gamma^{2(s-1)}} G(\eta, \rho, 0) = \frac{\partial^{2(s-1)}}{\partial \gamma^{2(s-1)}} F(\eta, \rho, 2\pi)$$

for $s \in \mathbb{N} \cap [1, p]$, $(\eta, \rho) \in [0, T] \times [0, 1]$. Applying the equalities (4.2) and (4.3) for $p = 2$, we obtain the uniformly convergence of the series $\sum_{n=0}^{\infty} S_n(\eta, r, \beta)$, by assumptions on F. Observe that the series $\sum_{m=1}^{\infty} h_{n,m}(\eta, r, \beta)$ is a Fourier–Bessel series which is uniformly convergent on $[\delta, 1 - \delta_1]$, with respect to the variable r. So, by proceeding as before, we prove the convergence of the series $\sum_{n=0}^{\infty} (\sum_{m=1}^{\infty} h_{n,m}(\eta, r, \beta))$.

5. The solution of the problem (F)

In order to construct a solution of the problem (F) given by (1.1)–(1.3) we shall prove two existence theorems.

Denote

$$(5.1) \qquad v_1(t, x) = v_1(t, r\cos\beta, r\sin\beta) = \widetilde{v}_1(t, r, \beta) = \\ = \frac{2}{\pi a^2} \int_0^a \Big[\int_0^{2\pi} G(t, r, \rho, \beta - \gamma) \widetilde{g}(\rho, \gamma)\, d\gamma \Big] \rho\, d\rho,$$

where $\widetilde{g}(\rho, \gamma) = g(\rho\cos\gamma, \rho\sin\gamma) = g(y)$, $y \in \Omega$, $(r\cos\beta, r\sin\beta) = x \in \Omega$, $t \in]0, T[$, the function g occurs in (1.2) and G is given by (3.3).

THEOREM 5.1. *If the function g fulfils the assumptions (2.2), then the function v_1 given by the formula (5.1) is a solution of the problem (F) with $\varphi = 0$.*

Proof. Making use of the properties of the function G and of the equality

$$G(t,r,\rho,\beta-\gamma)=\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{J_n(\mu_{n,m}\frac{r}{a})J_n(\mu_{n,m}\frac{\rho}{a})}{\varepsilon_n[J_{n+1}(\mu_{n,m})]^2}\times$$
$$\times(\cos n\beta\cos n\gamma+\sin n\beta\sin n\gamma)\exp\Big[-(\mu_{n,m})^2\frac{t}{a^2}\mathbb{A}\Big],$$

(see (3.3) and defining $A_{n,m}$, $C_{n,m}$ — the $\mathbb{R}^k$-valued coefficients of the Fourier–Bessel expansion of the function $\widetilde{g}$ by formulas

$$A_{n,m}=\frac{2}{\varepsilon_n\pi a^2[J_{n+1}(\mu_{n,m}]^2}\times\int_0^a\rho J_n\Big(\mu_{n,m}\frac{\rho}{a}\Big)\Big[\int_0^{2\pi}\cos n\gamma\widetilde{g}(\rho,\gamma)\,d\gamma\Big]\,d\rho,$$

$$C_{n,m}=\frac{2}{\varepsilon_n\pi a^2[J_{n+1}(\mu_{n,m})]^2}\times\int_0^a\rho J_n\Big(\mu_{n,m}\frac{\rho}{a}\Big)\Big[\int_0^{2\pi}\sin n\gamma\widetilde{g}(\rho,\gamma)\,d\gamma\Big]\,d\rho,$$

where $(n,m)\in J$, we can represent the function $\widetilde{v}_1$ in the form

$$\widetilde{v}_1(t,r,\beta)=\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\exp\Big[-(\mu_{n,m})^2\frac{t}{a^2}\mathbb{A}\Big]J_n\Big(\mu_{n,m}\frac{r}{a}\Big)\times$$
$$\times\{\cos n\beta A_{n,m}+\sin n\beta C_{n,m}\}.$$

Next, we are going to prove that the function $\widetilde{v}_1$ fulfils the equation

$$\frac{\partial\widetilde{v}_1}{\partial t}(t,r,\beta)=\mathbb{A}\widetilde{\Delta}\widetilde{v}_1(t,r,\beta). \tag{5.3}$$

Calculating derivatives of $\widetilde{v}_1$, we have

$$\frac{\partial\widetilde{v}_1}{\partial t}=\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\Big(-(\mu_{n,m})^2\frac{1}{a^2}\mathbb{A}\Big)\exp\Big[-(\mu_{n,m})^2\frac{t}{a^2}\mathbb{A}\Big]\times$$
$$\times\Big\{J_n\Big(\mu_{n,m}\frac{r}{a}\Big)\cos n\beta A_{n,m}+J_n\Big(\mu_{n,m}\frac{r}{a}\Big)\sin n\beta C_{n,m}\Big\},$$

$$\widetilde{\Delta}\widetilde{v}_1(t,r,\beta)=\frac{\partial^2\widetilde{v}_1}{\partial r^2}+\frac{1}{r}\frac{\partial\widetilde{v}_1}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v_1}{\partial\beta^2}=$$

$$= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \exp\left[-(\mu_{n,m})^2\frac{t}{a^2}\mathbb{A}\right]\Big\{\Big[\frac{d^2}{dr^2}J_n\Big(\mu_{n,m}\frac{r}{a}\Big)+$$

$$+\frac{1}{r}\frac{d}{dr}J_n\Big(\mu_{n,m}\frac{r}{a}\Big)-\frac{n^2}{r^2}J_n\Big(\mu_{n,m}\frac{r}{a}\Big)\Big]\cos n\beta A_{n,m}+$$

$$+\Big[\frac{d^2}{dr^2}J_n\Big(\mu_{n,m}\frac{r}{a}\Big)+\frac{1}{r}\frac{d}{dr}J_n\Big(\mu_{n,m}\frac{r}{a}\Big)+$$

$$-\frac{n^2}{r^2}J_n\Big(\mu_{n,m}\frac{r}{a}\Big)\Big]\sin n\beta C_{n,m}\Big\} =$$

$$= -\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \exp\left[-(\mu_{n,m})^2\frac{t}{a^2}\mathbb{A}\right]\times$$

$$\times\Big\{\Big(\frac{\mu_{n,m}}{a}\Big)^2 J_n\Big(\mu_{n,m}\frac{r}{a}\Big)(\cos n\beta A_{n,m}+\sin n\beta C_{n,m})\Big\}.$$

Since $W = J_n(\mu_{n,m}\frac{r}{a})$ is a solution of the Bessel equation

$$\frac{d^2W}{dr^2}+\frac{1}{r}\frac{dW}{dr}+(k_0^2-\frac{n^2}{r^2})W=0.$$

Thus the function $\widetilde{v}_1$ is a solution of the equation (5.3).

Now we can observe that

$$\widetilde{v}_1(0,r,\beta) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\mathbb{I}\Big\{J_n\Big(\mu_{n,m}\frac{r}{a}\Big)(\cos n\beta A_{n,m}+\sin n\beta C_{n,m})\Big\} = \widetilde{g}(r,\beta),$$

where $\mathbb{I}$ is the unit matrix $k\times k$. It follows from the definitions of $\widetilde{v}_1$ and $\mu_{n,m}$ that $\lim_{r\to a^-}\widetilde{v}_1(t,r,\beta) = 0 \in \mathbb{R}^k$. The proof of Theorem 5.1 is now complete.

Next, let us denote

(5.4) $v_2(t,x) = v_2(t, r\cos\beta, r\sin\beta) = \widetilde{v}_2(t,r,\beta) =$

$$= \frac{2}{\pi a^2}\int_0^t\Big[\int_0^{2\pi}\Big[\int_0^a \rho G(t-\tau,r,\rho,\beta-\gamma)\widetilde{\varphi}(\eta,\rho,\gamma)\,d\rho\Big]d\gamma\Big]d\eta,$$

where $\widetilde{\varphi}(t,r,\beta) = \varphi(t, r\cos\beta, r\sin\beta) = \varphi(t,x)$, $(r\cos\beta, r\sin\beta) = x\in\Omega$, $t\in\,]0,T[$.

THEOREM 5.2. *If the function φ fulfils the assumptions* (2.1), *then the function v_2 given by the formula* (5.4) *is a solution of the problem* (F) *with* $g = 0$.

Proof. Analogously as the function $\widetilde{g}$ in the proof of Theorem 5.1 we can represent the function $\widetilde{\varphi}$ for the fixed t in the form of the series

$$\widetilde{\varphi}(t,r,\beta) = \sum_{m=1}^{\infty} J_0\left(\mu_{0,m}\frac{r}{a}\right)A_{0,m}(t) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} J_n\left(\mu_{n,m}\frac{r}{a}\right)\times$$
$$\times\{\cos n\beta A_{n,m}(t) + \sin n\beta C_{n,m}(t)\}$$

where $A_{n,m}(t)$, $C_{n,m}(t)$ are the following $\mathbb{R}^k$-valued coefficients of the Fourier–Bessel expansion of the $\mathbb{R}^k$-valued function $\widetilde{\varphi}$ (for fixed t)

$$A_{n,m}(t) = \frac{2}{\varepsilon_n \pi a^2[J_{n+1}(\mu_{n,m})]^2}\times$$
$$\times \int_0^a \rho J_n\left(\mu_{n,m}\frac{\rho}{a}\right)\Big[\int_0^{2\pi}\widetilde{\varphi}(t,\rho,\gamma)\cos n\gamma\, d\gamma\Big]\, d\rho, \qquad (n,m)\in J,$$

$$C_{n,m}(t) = \frac{2}{\varepsilon_n \pi a^2[J_{n+1}(\mu_{n,m})]^2}\times$$
$$\times \int_0^a \rho J_n\left(\mu_{n,m}\frac{\rho}{a}\right)\Big[\int_0^{2\pi}\widetilde{\varphi}(t,\rho,\gamma)\sin n\gamma\, d\gamma\Big]\, d\rho, \qquad (n,m)\in J.$$

Making use of the properties of G, we can represent the function $\widetilde{v}_2$ in the form

$$\widetilde{v}_2(t,r,\beta) =$$
$$= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\Big\{\cos n\beta J_n\left(\mu_{n,m}\frac{r}{a}\right)\int_0^t \exp\Big[-\left(\mu_{n,m}\frac{1}{a}\right)^2(t-\eta)\mathbb{A}\Big]A_{n,m}(t)\, d\eta +$$
$$+ \sin n\beta J_n\left(\mu_{n,m}\frac{r}{a}\right)\int_0^t \exp\Big[-\left(\mu_{n,m}\frac{1}{a}\right)^2(t-\eta)\mathbb{A}\Big]C_{n,m}(t)\, d\eta\Big\}.$$

Calculating derivatives of $\widetilde{v}_2$ we obtain

$$\frac{\partial}{\partial t}\widetilde{v}_2(t,r,\beta) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\Big\{\cos n\beta J_n\left(\mu_{n,m}\frac{r}{a}\right)A_{n,m}(t) +$$
$$+ \sin n\beta J_n\left(\mu_{n,m}\frac{r}{a}\right)C_{n,m}(t)\Big\} +$$

$$+\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n\left(\mu_{n,m}\frac{r}{a}\right)\left(-\left(\mu_{n,m}\frac{1}{a}\right)^2\mathbb{A}\right)\times$$

$$\times\left[\cos n\beta\int_0^t \exp\left[-\left(\mu_{n,m}\frac{1}{a}\right)^2(t-\eta)\mathbb{A}\right]A_{n,m}(t)\,d\eta+\right.$$

$$\left.+\sin n\beta\int_0^t \exp\left[-\left(\mu_{n,m}\frac{1}{a}\right)^2(t-\eta)\mathbb{A}\right]C_{n,m}(t)\,d\eta\right],$$

$$\widetilde{\Delta}\widetilde{v}_2(t,r,\beta)=$$

$$=\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left[\frac{d^2}{dr^2}J_n\left(\mu_{n,m}\frac{r}{a}\right)+\frac{1}{r}\frac{d}{dr}J_n\left(\mu_{n,m}\frac{r}{a}\right)-\left(\frac{n}{r}\right)^2J_n\left(\mu_{n,m}\frac{r}{a}\right)\right]\times$$

$$\times\left[\cos n\beta\int_0^t \exp\left[-\left(\mu_{n,m}\frac{1}{a}\right)^2(t-\eta)\mathbb{A}\right]A_{n,m}(t)d\eta+\right.$$

$$\left.+\sin n\beta\int_0^t \exp\left[-\left(\mu_{n,m}\frac{1}{a}\right)^2(t-\eta)\mathbb{A}\right]C_{n,m}(t)\,d\eta\right]=$$

$$=-\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left[\left\{\left(\mu_{n,m}\frac{1}{a}\right)^2\cos n\beta J_n\left(\mu_{n,m}\frac{r}{a}\right)\times\right.\right.$$

$$\left.\times\int_0^t \exp\left[-\left(\mu_{n,m}\frac{1}{a}\right)^2(t-\eta)\mathbb{A}\right]A_{n,m}(t)\,d\eta\right\}+$$

$$+\left\{\left(\frac{\mu_{n,m}}{a}\right)^2\sin n\beta J_n\left(\mu_{n,m}\frac{r}{a}\right)\times\right.$$

$$\left.\left.\times\int_0^t \exp\left[-\left(\mu_{n,m}\frac{1}{a}\right)^2(t-\eta)\mathbb{A}\right]C_{n,m}(t)\,d\eta\right\}\right],$$

since the function $J_n(\mu_{n,m}\frac{r}{a})$ is a solution of the Bessel equation, analogously as in the proof of Theorem 5.1. Thus, we obtain

$$\frac{\partial}{\partial t}\widetilde{v}_2(t,r,\beta)=\mathbb{A}\widetilde{\Delta}\widetilde{v}_2(t,r,\beta)+\widetilde{\varphi}(t,r,\beta).$$

From (5.4) it follows that $\widetilde{v}_2(0,r,\beta)=0\in\mathbb{R}^k$ and $\lim_{r\to a^-}\widetilde{v}_2(t,r,\beta)=0\in\mathbb{R}^k$.

COROLLARY 5.1. *Suppose that the functions g and φ fulfil the assumptions* (2.2) *and* (2.1), *respectively. Then the function $v = v_1 + v_2$ is a solution of the problem* (F).

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A CLASS OF MULTIVALENT FUNCTIONS WITH NEGATIVE TAYLOR COEFFICIENTS

1. Introduction

Let D denote the unit disc $\{z : |z| < 1\}$. Let A_p, $p = 1, 2, \ldots$, be the class of functions f analytic in D and represented by the Taylor series

$$(1.1) \qquad f(z) = z^p + \sum_{k=1}^{\infty} a_k z^{k+p}, \quad z \in D.$$

A function f in A_p is said to be p-valent starlike of order α, $0 \leq \alpha < 1$, if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > pa, \quad z \in D,$$

and f is said to be a p-valent convex function of order α, if $\frac{zf'(z)}{p}$ is p-valent starlike of order α [1].

Let $S^{\diamond}[p, \alpha]$ denote the class of p-valent starlike functions f of order α, given by the Taylor series

$$(1.2) \qquad f(z) = z^p - \sum_{k=1}^{\infty} |a_k| z^{k+p}, \quad z \in D.$$

Similarly, let $K[p, \alpha]$ denote the class of p-valent convex functions of order α that are represented by (1.2). Kapoor and Mishra [3] have proved that a

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function f given by (1.2) is in $S^\circ[p,\alpha]$ if and only if

$$\sum_{k=1}^{\infty}(k+p-p\alpha)|a_k| \le p(1-\alpha) \tag{1.3}$$

and f is in $K[p,\alpha]$ if and only if

$$\sum_{k=1}^{\infty}\left(\frac{k+p}{p}\right)(k+p-p\alpha)|a_k| \le p(1-\alpha). \tag{1.4}$$

For any real number λ, let $S_\lambda[p,\alpha]$, $0 \le \alpha < 1$, denote the class of functions f given by the Taylor series (1.2) and satisfying

$$\sum_{k=1}^{\infty}\left(\frac{k+p}{p}\right)^{\lambda}(k+p-p\alpha)|a_k| \le p(1-\alpha). \tag{1.5}$$

It follows that $S_0[p,\alpha] = S^\circ[p,\alpha]$, $S_1[p,\alpha] = K[p,\alpha]$ and if $\lambda_1 < \lambda_2$, then $S_{\lambda_2}[p,\alpha] \subsetneq S_{\lambda_1}[p,\alpha]$. Thus, for $\lambda \ge 0$, functions in $S_\lambda[p,\alpha]$ are p-valent starlike of order α and further if $\lambda \ge 1$, functions in $S_\lambda[p,\alpha]$ are p-valent convex of order α. For $\lambda \ge 0, p = 1$, this class has been studied by Kumar [4]. For $\lambda < 0$, $S_\lambda[p,\alpha]$ contains some non p-valent functions also.

Example. Consider the functions f_k in $S_\lambda[p,\alpha]$ given by

$$f_k(z) = z^p - \frac{p^{\lambda+1}(1-\alpha)}{(k+p-p\alpha)(k+p)^{\lambda}}z^{k+p}, \quad z \in D, \tag{1.6}$$

where $0 \le \alpha < 1$, $k,p = 1,2,\dots$ and $\lambda < 0$. We shall show that, for fixed α and p, there exists a positive integer k_0 such that for all $k > k_0$, the pth derivative of each function f_k vanishes at some point in D, so that f_k is not p-valent for $k > k_0$.

In fact, $f_k^{(p)}(z)$ is equal to zero, if

$$z^k = \frac{p-1}{k+p-1}\,\frac{p-2}{k+p-2}\cdots\frac{1}{k+1}\,\frac{p^{-\lambda}(k+p-p\alpha)}{(k+p)^{1-\lambda}(1-\alpha)}.$$

Thus, $|z| < 1$ is satisfied if

$$\frac{p^{-\lambda}(k+p-p\alpha)}{(k+p)^{1-\lambda}(1-\alpha)} < 1$$

which is equivalent to

$$\alpha < \frac{(k+p)^{1-\lambda} - p^{-\lambda}(k+p)}{(k+p)^{1-\lambda} - p^{1-\lambda}} = F(k). \tag{1.7}$$

Note that $F(k)$ is increasing for $k > (p/(-\lambda))$ and $\lim_{k\to\infty} F(k) = 1$. Thus, for fixed α and p, we can choose a k_0 such that (1.7) is satisfied for all

$k > k_0 > (p/(-\lambda))$. The function $F(k)$ is increasing what can be seen from the following calculations. $F(k) < F(k+1)$ is equivalent to

$$\left(1+\frac{1}{k+p}\right)^{1-\lambda} - \left(1+\frac{1}{k}\right) + \frac{p^{1-\lambda}}{k(k+p)^{1-\lambda}} > 0. \tag{1.8}$$

Since

$$\left(1+\frac{1}{k+p}\right)^{1-\lambda} > 1 + \frac{1-\lambda}{k+p}$$

it follows that (1.8) is true if $k > (p/(-\lambda))$.

In the present paper we study the class $S_\lambda[p,\alpha]$ for all nonnegative as well as for some negative values of λ. In particular, we obtain distortion theorems, a covering theorem, order of starlikeness and order of convexity for the family $S_\lambda[p,\alpha]$. These results extend those of Silverman in [5]. Our result on quasi-Hadamard product of several functions in $S_\lambda[p,\alpha]$ unifies and generalizes some recent results of Kumar [4].

2. Distortion and covering theorems

It follows from (1.5) that if f given by (1.2) is in $S_\lambda[p,\alpha]$, then

$$|a_k| \le \frac{(1-\alpha)p^{\lambda+1}}{(k+p-p\alpha)(k+p)^\lambda}. \tag{2.1}$$

Equality holds in (2.1), for each $k = 1, 2, \dots$, only for the functions

$$f_k(z) = z^p - \frac{(1-\alpha)p^{\lambda+1}}{(k+p-p\alpha)(k+p)^\lambda} z^{k+p}, \quad z \in D. \tag{2.2}$$

The function

$$f_1(z) = z^p - \frac{(1-\alpha)p^{\lambda+1}}{(1+p-p\alpha)(1+p)^\lambda} z^{p+1}, \quad z \in D, \tag{2.3}$$

is of foremost importance in the discussion of sharpness for the results of this paper.

We obtain the following results.

THEOREM 1. *Let f be in $S_\lambda[p,\alpha]$. Then for $\lambda \ge -1$*

$$|f(z)| \le r^p + \frac{(1-\alpha)p^{\lambda+1}}{(1+p-p\alpha)(1+p)^\lambda} r^{1+p}, \tag{2.4}$$

$$|f(z)| \ge r^p - \frac{(1-\alpha)p^{\lambda+1}}{(1+p-p\alpha)(1+p)^\lambda} r^{1+p}, \tag{2.5}$$

where $r = |z|$. Equality holds in (2.4) at $z = -r$ and in (2.5) at $z = r$ only for the function f_1 defined by (2.3).

Proof. For $\lambda \geq -1$, $(\frac{k+p}{p})^{\lambda}(k+p-p\alpha)$ is an increasing sequence in k. So,

$$p(1-\alpha) \geq \sum_{k=1}^{\infty} \left(\frac{k+p}{p}\right)^{\lambda} (k+p-p\alpha)|a_k|$$
$$\geq \left(\frac{1+p}{p}\right)^{\lambda} (1+p-p\alpha) \sum_{k=1}^{\infty} |a_k|.$$

Equivalently,

$$\sum_{k=1}^{\infty} |a_k| \leq \frac{(1-\alpha)p^{\lambda+1}}{(1+p-p\alpha)(1+p)^{\lambda}}. \tag{2.6}$$

Using (2.6), we have

$$|f(z)| \leq |z|^p + \sum_{k=1}^{\infty} |a_k|\,|z|^{k+p} \leq |z|^p + |z|^{1+p} \sum_{k=1}^{\infty} |a_k|$$
$$\leq |z|^p + |z|^{1+p} \left(\frac{p}{1+p}\right)^{\lambda} \frac{p(1-\alpha)}{(1+p-p\alpha)}.$$

Hence we get (2.4). Similarly

$$|f(z)| \geq |z|^p - \sum_{k=1}^{\infty} |a_k|\,|z|^{k+p} \geq |z|^p - |z|^{1+p} \sum_{k=1}^{\infty} |a_k|$$

and, by (2.6), we obtain (2.5).

It is clear that equality holds in (2.4) and in (2.5) only if it holds in (2.6). However, in view of (1.5), this is true only if

$$|a_1| = \frac{(1-\alpha)p^{\lambda+1}}{(1+p-p\alpha)(1+p)^{\lambda}}. \tag{2.7}$$

Thus, as indicated at the beginning of this section, f must be equal to f_1 defined by (2.3). The proof is complete.

THEOREM 2. *For $-1 \leq \lambda < 0$, let*

$$\alpha_0 = 1 - \frac{p^{-\lambda}}{(1+p)^{1-\lambda} - p^{1-\lambda}}.$$

The disc $|z| < 1$ is mapped by any function in $S_{\lambda}[p,\alpha]$, $\lambda \geq -1$, onto a domain that contains the disc

(a) $|w| < 1 - \frac{(1-\alpha)p^{\lambda+1}}{(1+p-p\alpha)(1+p)^{\lambda}} \equiv r_1(p,\alpha,\lambda)$, *if $\lambda \geq 0$ and $0 \leq \alpha < 1$ or $-1 \leq \lambda < 0$ and $\alpha_0 < \alpha < 1$,*

(b) $|w| < \frac{1}{p+1}[\frac{1+p-p\alpha}{(1+p)(1-\alpha)}(\frac{1+p}{p})^{\lambda}]^p \equiv r_2(p,\alpha,\lambda)$, *if $-1 \leq \lambda < 0$ and $0 \leq \alpha < \alpha_0$.*

Proof. First, note that, if $\lambda < 0$, then

$$0 < \frac{p^{-\lambda}}{(1+p)^{1-\lambda} - p^{1-\lambda}} < 1.$$

Therefore, $0 < \alpha_0 < 1$. In view of Theorem 1 (2.5), the range of f contains the disc $|w| < \max(g(r))$, $0 < r < 1$, where

$$g(r) = r^p - \frac{p(1-\alpha)}{(1+p-p\alpha)}\left(\frac{p}{p+1}\right)^{\lambda} r^{p+1}.$$

Differentiation with respect to r gives

$$g'(r) = pr^{p-1}\left[1 - \frac{(1+p)(1-\alpha)}{(1+p-p\alpha)}\left(\frac{p}{p+1}\right)^{\lambda} r\right].$$

Note that $[(1+p)(1-\alpha)/(1+p-p\alpha)] < 1$, if $\alpha > 0$, and $(p/(1+p))^{\lambda} \leq 1$, if $\lambda \geq 0$. Therefore, g is increasing in r, $0 < r < 1$, if $0 < \alpha < 1$ and $\lambda \geq 0$, so $\max(g(r)) = g(1)$. This gives the first part of (a).

Next, if $-1 \leq \lambda < 0$, then the condition

$$\frac{(1+p)(1-\alpha)}{(1+p-p\alpha)}\left(\frac{p}{1+p}\right)^{\lambda} \leq 1$$

is equivalent to $\alpha_0 \leq \alpha$. Therefore, if $-1 \leq \lambda < 0$ and $\alpha_0 \leq \alpha < 1$, then g is increasing in r, $0 < r < 1$. Thus $\max(g(r)) = g(1)$. This gives the second part of (a).

On the other hand, if $-1 \leq \lambda < 0$ and $0 \leq \alpha < \alpha_0$, then $g'(r_0) = 0$, where

$$r_0 = \left(\frac{p+1}{p}\right)^{\lambda} \frac{(1+p-p\alpha)}{(1+p)(1-\alpha)} < 1,$$

and

$$g''(r_0) = -p\left[\frac{(1+p-p\alpha)}{(1+p)(1-\alpha)}\left(\frac{1+p}{p}\right)^{\lambda}\right]^{p-2} < 0.$$

Therefore,

$$\max g(r) = g(r_0) = \frac{1}{(p+1)}\left[\frac{(1+p-p\alpha)}{(1+p)(1-\alpha)}\left(\frac{p+1}{p}\right)^{\lambda}\right]^{p}.$$

This gives (b). The proof is complete.

Remark 1. By choosing $p = 1$, $\lambda = 0$ and $p = 1$, $\lambda = 1$ in Theorem 2 we get

$$r_1(1,\alpha,0) = \frac{1}{2-\alpha} \quad \text{and} \quad r_1(1,\alpha,1) = \frac{3-\alpha}{4-2\alpha},$$

a result of Silverman [5, Theorem 5]. Further, for fixed p and λ, $-1 \leq \lambda \leq 0$, a routine calculation shows that

$$\frac{1}{(p+1)} = r_1(p, \alpha_0, \lambda) = \lim_{\alpha \to \alpha_0} r_2(p, \alpha, \lambda).$$

THEOREM 3. *For $f \in S_\lambda[p, \alpha]$, $0 \leq \alpha < 1$, $\lambda \geq 0$, we have*

$$|f'(z)| \leq pr^{p-1}\left(1 + \frac{p^\lambda(1-\alpha)}{(1+p)^{\lambda-1}(1+p-p\alpha)}r\right), \tag{2.8}$$

$$|f'(z)| \geq pr^{p-1}\left(1 - \frac{p^\lambda(1-\alpha)}{(1+p)^{\lambda-1}(1+p-p\alpha)}r\right), \tag{2.9}$$

with $r = |z|$. Equality holds in (2.8) at $z = -r$ and in (2.9) at $z = r$ only for the function f_1 defined by (2.3).

Proof. Since

$$(1+p-p\alpha)\sum_{k=1}^{\infty}\left(\frac{k+p}{p}\right)^\lambda |a_k| \leq \sum_{k=1}^{\infty}\left(\frac{k+p}{p}\right)^\lambda (k+p-p\alpha)|a_k| \leq p(1-\alpha),$$

we have

$$\sum_{k=1}^{\infty}\left(\frac{k+p}{p}\right)^\lambda |a_k| \leq \frac{p(1-\alpha)}{(1+p-p\alpha)}. \tag{2.10}$$

Using (1.5) and (2.10), we get

$$\begin{aligned}\left(\frac{1+p}{p}\right)^\lambda \sum_{k=1}^{\infty}(k+p)|a_k| &\leq \sum_{k=1}^{\infty}\left(\frac{k+p}{p}\right)^\lambda (k+p)|a_k| \\ &= \sum_{k=1}^{\infty}\left(\frac{k+p}{p}\right)^\lambda (k+p-p\alpha)|a_k| + \sum_{k=1}^{\infty}\left(\frac{k+p}{p}\right)^\lambda p\alpha|a_k| \\ &\leq p(1-\alpha) + p\alpha\frac{p(1-\alpha)}{(1+p-p\alpha)} = \frac{p(1+p)(1-\alpha)}{(1+p-p\alpha)}.\end{aligned}$$

So,

$$\sum_{k=1}^{\infty}(k+p)|a_k| \leq \frac{p^{\lambda+1}}{(1+p)^{\lambda-1}}\frac{(1-\alpha)}{(1+p-p\alpha)}. \tag{2.11}$$

Using (2.11), we get for $|z| = r$

$$\begin{aligned}|f'(z)| &\leq p|z|^{p-1} + \sum_{k=1}^{\infty}(k+p)|a_k||z|^{k+p-1} \\ &\leq pr^{p-1} + r^p\sum_{k=1}^{\infty}(k+p)|a_k|\end{aligned}$$

$$\leq pr^{p-1} + r^p \frac{p^{\lambda+1}}{(1+p)^{\lambda-1}} \frac{(1-\alpha)}{(1+p-p\alpha)}$$
$$= pr^{p-1}\left(1 + \frac{(1-\alpha)p^\lambda}{(1+p)^{\lambda-1}(1+p-p\alpha)}r\right).$$

This gives (2.8). Similarly,

$$|f'(z)| \geq p|z|^{p-1} - \sum_{k=1}^{\infty}(k+p)|a_k||z|^{k+p-1}$$
$$\geq pr^{p-1}\left(1 - \frac{(1-\alpha)p^\lambda}{(1+p-p\alpha)(1+p)^{\lambda-1}}r\right).$$

This gives (2.9).

We note that equality in (2.10), hence in (2.11), holds only if (2.7) is true. So equality holds in (2.8) at $z = -r$ and in (2.9) at $z = r$ only for the function f_1 defined by (2.3). This completes the proof.

3. Quasi-Hadamard product

Let f and g be two functions analytic in D and be represented by the Taylor series

$$f(z) = a_0 z^p - \sum_{k=1}^{\infty} a_k z^{k+p}, \quad a_0 > 0,\ a_k \geq 0,\ k = 1, 2, \ldots,$$
$$g(z) = b_0 z^p - \sum_{k=1}^{\infty} b_k z^{k+p}, \quad b_0 > 0,\ b_k \geq 0,\ k = 1, 2, \ldots.$$

The quasi-Hadamard product of f and g is defined to be the analytic function $f \diamond g$ given by the Taylor series

$$(f \diamond g)(z) = a_0 b_0 z^p - \sum_{k=1}^{\infty} a_k b_k z^{k+p}.$$

Note that the usual Hadamard product would give the Taylor series

$$a_0 b_0 z^p + \sum_{k=1}^{\infty} a_k b_k z^{k+p}.$$

For $p = 1$, the above definition of quasi-Hadamard product is due to Kumar [4]. Quasi-Hadamard product of several functions is defined similarly. We next prove a theorem that generalizes and unifies some recent results of Kumar [4, Theorem A, B, C].

THEOREM 4. *For $i = 1, 2, \ldots, m$ let g_i, given by*

$$g_i(z) = z^p - \sum_{k=1}^{\infty} a_{k,i} z^{k+p}, \quad a_{k,i} \geq 0, \ k = 1, 2, \ldots,$$

be in $S_{\lambda_i}[p, \alpha_i]$, $0 \leq \alpha_i < 1$ and $-\infty < \lambda_i < \infty$. Then, the function $g = g_1 \diamond g_2 \diamond \ldots \diamond g_m$ is in $S_\mu[p, \alpha^\diamond]$, where $\mu = \sum_{i=1}^{m} \lambda_i + (m-1)$ and $\alpha^\diamond = \max\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$.

Proof. It is sufficient to show that

$$\sum_{k=1}^{\infty} \left[\left(\frac{k+p}{p} \right)^{\mu} (k + p - p\alpha^\diamond) \prod_{i=1}^{m} a_{k,i} \right] \leq p(1 - \alpha^\diamond).$$

Without loss of generality we assume that $\alpha_m = \alpha^\diamond$. Since $g_i \in S_{\lambda_i}[p, \alpha_i]$, we have

$$(3.1) \qquad \sum_{k=1}^{\infty} \left(\frac{k+p}{p} \right)^{\lambda_i} (k + p - p\alpha_i) a_{k,i} \leq p(1 - \alpha_i)$$

for every $i = 1, 2, \ldots, m$.

Therefore,

$$(3.2) \qquad a_{k,i} \leq \frac{p(1-\alpha_i)}{k + p - p\alpha_i} \left(\frac{p}{k+p} \right)^{\lambda_i} \leq \left(\frac{p}{k+p} \right)^{\lambda_i + 1} .$$

Using (3.2) for $i = 1, 2, \ldots, m-1$ and (3.1) for $i = m$, we obtain,

$$\sum_{k=1}^{\infty} \left[\left(\frac{k+p}{p} \right)^{\mu} (k + p - p\alpha^\diamond) \left(\prod_{i=1}^{m} a_{k,i} \right) \right]$$

$$\leq \sum_{k=1}^{\infty} \left[\left(\frac{k+p}{p} \right)^{\mu} (k + p - p\alpha^\diamond) \left(\frac{p}{k+p} \right)^{\lambda_1 + \ldots + \lambda_{m-1} + m - 1} a_{k,m} \right]$$

$$= \sum_{k=1}^{\infty} \left[\left(\frac{k+p}{p} \right)^{\lambda_m} (k + p - p\alpha_m) a_{k,m} \right] \leq p(1 - \alpha_m) = p(1 - \alpha^\diamond).$$

Hence, $g = g_1 \diamond g_2 \diamond \ldots \diamond g_m \in S_\mu[p, \alpha^\diamond]$. The proof is complete.

Remark 2. Theorem 4 is new even in the case of $p = 1$ and provides several interesting consequences. If we take $\lambda_i = 0$ and $\lambda_i = 1$ for $i = 1, 2, \ldots, m$ in Theorem 4 with $p = 1$, we get Theorem A and Theorem B of [4], respectively. Similarly, the choice of $\lambda_i = 0$ for $i = 1, 2, \ldots, q$ and $\lambda_i = 1$ for $i = q+1, q+2, \ldots, m$ in Theorem 4 with $p = 1$, gives Theorem C of [4].

A general problem in the theory of univalent functions concerns the study of those transformations which carry one or several univalent as well

as nonunivalent functions into the class of univalent functions. A useful consequence of Theorem 4, related to the above problem is that quasi-Hadamard product of suitable number of nonunivalent functions can produce a univalent function. For example, in Theorem 4 with $p = 1$, if we choose $-1 + \frac{1}{m} \le \lambda_i \le 0$, $i = 1, 2, \ldots, m$, then the quasi-Hadamard product $f_1 \diamond f_2 \diamond \ldots \diamond f_m$ of functions f_i in $S_{\lambda_i}[1, \alpha]$ is a starlike univalent function. Similarly, if $m \ge 2$, $i = 1, 2, \ldots, m$, and $\frac{2}{m} - 1 \le \lambda_i \le 0$, then $f_1 \diamond f_2 \diamond \ldots \diamond f_m$, for f_i in $S_{\lambda_i}[1, \alpha]$, is a convex univalent function. A yet simpler example gives that the quasi-Hadamard product of an univalent convex function given by (1.2) and a function in $S_{-1}[1, \alpha]$ is a convex univalent function. Similar remarks for p-valent cases also follow from Theorem 4.

4. Order of starlikeness and order of convexity for $S_\lambda[p, \alpha]$

Silverman [4] has shown that the order of starlikeness of the family of univalent convex functions of order α is equal to $\frac{2}{3-\alpha}$. Kapoor and Mishra [3] have shown that, if f given by (1.2) is in $K[p, \alpha]$, then f is in $S^\circ[p, \beta]$, where

$$\beta \equiv \beta(\alpha) = \frac{p+1}{2p+1-p\alpha}.$$

THEOREM 5. *Suppose $-\infty < t < s < \infty$, $s - t \ge 1$, $p = 1, 2, \ldots$, and $1 - \frac{1}{p^2} \le \alpha < 1$. If $f \in S_s[p, \alpha]$, then $f \in S_t[p, \beta]$, where*

$$\beta = \frac{(\frac{1+p}{p})^m (1 + p - p\alpha) - (1+p)(1-\alpha)}{(\frac{1+p}{p})^m (1 + p - p\alpha) - p(1-\alpha)}, \quad m = s - t. \tag{4.1}$$

The result is sharp with

$$f(z) = z^p - \left(\frac{p}{1+p}\right)^s \frac{p(1-\alpha)}{(1+p-p\alpha)} z^{p+1} \tag{4.2}$$

being extremal.

Proof. In view of (1.5), it suffices to show that

$$\frac{(\frac{k+p}{p})^t (k + p - p\beta)}{1-\beta} \le \frac{(\frac{k+p}{p})^s (k + p - p\alpha)}{1-\alpha}$$

for all $k = 1, 2, \ldots$. This is equivalent to show that

$$\beta \le \frac{(\frac{k+p}{p})^s (k + p - p\alpha) - (\frac{k+p}{p})^t (k+p)(1-\alpha)}{(\frac{k+p}{p})^s (k + p - p\alpha) - p(\frac{k+p}{p})^t (1-\alpha)}.$$

Equivalently

$$\beta \le \frac{(\frac{k+p}{p})^m (k + p - p\alpha) - (k+p)(1-\alpha)}{(\frac{k+p}{p})^m (k + p - p\alpha) - p(1-\alpha)} = F(k), \quad k = 1, 2, \ldots.$$

Since $F(k)$ is an increasing sequence of k for $s-t \geq 1$ and $1-\frac{1}{p^2} \leq \alpha < 1$, the above is true, if we take $\beta = F(1)$. In order to show that F is an increasing sequence of k, extend F as a function of the real variable x. We get that $F'(x)$ is equal to the expression given below

$$(4.3) \quad \begin{aligned} &\{(x+p)^{s+t-1}p^{s+t}[(s-t)(x+p)^2 - p(\alpha(s-t-1)+s-t+1)(x+p) \\ &+ \alpha p^2(s-t)] + (1-\alpha)p^{2s+1}(x+p)^{2t}\}(1-\alpha)[p^t(x+p)^{s+1} \\ &- \alpha p^{t+1}(x+p)^s - p^{s+1}(x+p)^t + \alpha p^{s+1}(x+p)^t]^{-2}. \end{aligned}$$

Put $g(r) = (s-t)r^2 - p(\alpha(s-t-1)+s-t+1)r + \alpha p^2(s-t)$. Then

$$\begin{aligned} g(0) &= \alpha p^2(s-t) > 0, \\ g(p) &= -p^2(1-\alpha) < 0, \\ g(1+p) &= (s-t)(1+p-p\alpha) - p(1+p)(1-\alpha). \end{aligned}$$

Since $s-t \geq 1$ and $\alpha \geq 1-\frac{1}{p^2}$, we get $g(1+p) > 0$, so both the roots of $g(r)$ lie in the interval $(0, 1+p)$ and it follows that $g(r) > 0$ for $r \geq 1+p$.

This gives that $F'(x) > 0$ for $x \geq 1$ and $F(k)$ in an increasing sequence in k.

A calculation gives that the $(p+1)$ coefficient of the function f in $S_s[p,\alpha]$ given by (4.2) satisfies

$$\left(\frac{1+p}{p}\right)^t \frac{(1+p-p\beta)}{p(1-\beta)} \left(\frac{p}{1+p}\right)^s \frac{p(1-\alpha)}{(1+p-p\alpha)} = 1,$$

where β is given by (4.1). Therefore, in view of (1.5), f is in $S_t[p,\beta]$. Hence the value of β given by (4.1) is the largest.

Remark 3. Theorem 5 has many interesting consequences in the case of $p = 1$ also.

(a) Taking $t = 0$ and $s \geq 1$, we get that the order of (univalent) starlikeness of any function in $S_s[1,\alpha] \equiv S_s[\alpha]$, $0 \leq \alpha < 1$, is equal to

$$(4.4) \qquad \frac{2^s(2-\alpha) - 2(1-\alpha)}{2^s(2-\alpha) - (1-\alpha)}.$$

The choice $s = 1$ in (4.4) gives the result of Silverman [5, Theorem 7].

(b) Taking $t = 1$ and $s \geq 2$, we get that the order of (univalent) convexity for any function in $S_s[1,\alpha] = S_s[\alpha]$, $0 \leq \alpha < 1$, is equal to

$$(4.5) \qquad \frac{2^s(2-\alpha) - 4(1-\alpha)}{2^s(2-\alpha) - 2(1-\alpha)}.$$

(c) Taking $t = 0$ and $s \geq 1$, we get that the order of p-valent starlikeness of any function in $S_s[p, \alpha]$, $1 - \frac{1}{p^2} \leq \alpha < 1$, is equal to

$$(4.6) \qquad \frac{(\frac{1+p}{p})^s(1 + p - p\alpha) - (1 + p)(1 - \alpha)}{(\frac{1+p}{p})^s(1 + p - p\alpha) - p(1 - \alpha)}.$$

The choice $s = 1$ in (4.6) gives the result of Kapoor and Mishra [3, Theorem 4(i)].

(d) Taking $t = 1$ and $s \geq 2$, we get that the order of p-valent convexity for any function in $S_s[p, \alpha]$, $1 - \frac{1}{p^2} \leq \alpha < 1$, is equal to

$$(4.7) \qquad \frac{(\frac{1+p}{p})^s(1 + p - p\alpha) - (\frac{1+p}{p})(1 + p)(1 - \alpha)}{(\frac{1+p}{p})^s(1 + p - p\alpha) - (\frac{1+p}{p})p(1 - \alpha)}.$$

In our next theorem we determine the sharp value of a constant $\gamma \equiv \gamma(\alpha)$ such that

$$\operatorname{Re}\left\{\frac{f(z)}{z^p}\right\} > p\gamma$$

for f in $S_\lambda[p, \alpha]$, $\lambda \geq -1$ and $z \in D$. We note that in general, if we do not assume the coefficients to be negative, then there does not exist any positive constant γ such that

$$\operatorname{Re}\left\{\frac{f(z)}{z^p}\right\} > p\gamma$$

for p-valent convex functions of order α [2].

THEOREM 6. *Let f, given by the Taylor series* (1.2), *be in $S_\lambda[p, \alpha]$, for $\lambda \geq -1$. Then*

$$(4.8) \qquad \operatorname{Re}\left\{\frac{f(z)}{z^p}\right\} > p\gamma,$$

where

$$(4.9) \qquad \gamma \equiv \gamma(\alpha) = 1 - \left(\frac{p}{1+p}\right)^\lambda \frac{(1-\alpha)}{(1 + p - p\alpha)}.$$

The above result is sharp, the function f_1 defined by (2.3) *being extremal.*

Proof. Kapoor and Mishra [3] have shown that a function f given by (1.2) satisfies (4.8) if and only if

$$(4.10) \qquad \sum_{k=1}^{\infty} \frac{|a_k|}{p(1-\gamma)} \leq 1.$$

Since f is in $S_\lambda[p, \alpha]$, we have

$$\sum_{k=1}^{\infty}\left(\frac{k+p}{p}\right)^{\lambda}\frac{(k+p-p\alpha)}{p(1-\alpha)}|a_k| \leq 1.$$

Thus, it is sufficient to show that

$$\frac{1}{p(1-\gamma)} \leq \left(\frac{k+p}{p}\right)^{\lambda}\frac{(k+p-p\alpha)}{p(1-\alpha)}, \quad \text{for all } k = 1, 2, \ldots.$$

This is equivalent to

$$\gamma \leq 1 - \left(\frac{p}{k+p}\right)^{\lambda}\frac{(1-\alpha)}{(k+p-p\alpha)} = \phi(k), \quad k = 1, 2, \ldots.$$

A brief calculation shows that $\phi(k)$ is an increasing sequence in k, if $\lambda \geq -1$. Thus we get (4.9) by taking $\gamma = \phi(1)$. We note that

$$\frac{(1-\alpha)p^{\lambda+1}}{(1+p-p\alpha)(1+p)^{\lambda}}\frac{1}{p(1-\gamma)} = 1,$$

where γ is given by (4.9). Therefore, in view of (4.10), the value of γ defined by (4.9) is the largest. This completes the proof.

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ON TRANSLATE OF BERNSTEIN TYPE RATIONAL POLYNOMIALS

1. Introduction

Pàpanicolau [2] studied some approximation results on bounded continuous functions f by a class of linear operators $(L_{n,t}f)$ defined as

$$(1.1)\qquad (L_{n,t}f)(x) = \sum_{k=0}^{n} \binom{k+n-1}{k} \frac{t^k}{(1+t)^{n+k}} f\left(x+\frac{k}{n}\right), \qquad \text{for } f \in C_B[0,\infty).$$

Now, following the operators (1.1), we define the following Bernstein type rational polynomials $(P_{n,t}f)$ as

$$(1.2)\qquad (P_{n,t}f)(x) = \sum_{k=0}^{n} A_{n,k,t} f\left(x+\frac{k}{n^{\alpha}}\right),$$

where

$$(1.3)\qquad A_{n,k,t} = \binom{n}{k}\left(\frac{n^{\alpha-1}t)^k}{(1+n^{\alpha-1}t)^n}\right.$$

$$f \in C_B[0,\infty) \quad \text{and} \quad a \in (0,1],$$

and study some approximation results on the operators (1.2).

2. In this section we prove some basic results which are useful in proving the main results.

LEMMA. *For $t \geq 0$ and $n \in \mathbb{N}$, the following identities hold*

$$(2.1)\qquad \sum_{k=0}^{n} A_{n,k,t} = t,$$

$$(2.2)\qquad \sum_{k=0}^{n} A_{n,k,t}\left(\frac{k}{n^\alpha}-t\right) = -\frac{n^{\alpha-1}t^2}{1+n^{\alpha-1}t},$$

$$(2.3)\qquad \sum_{k=0}^{n} A_{n,k,t}\left(\frac{k}{n^\alpha}-t\right)^2 = \frac{(n^{2\alpha-2}t^4+n^{-\alpha}t)}{(t+n^{\alpha-1}t)^2} = B_{n,\alpha,t}\ (say).$$

Proof. On differentiating the expression (2.1) with respect to t and adjusting the terms, we get the required results (2.2) and (2.3). However the proof is similar to that of K. Balâzs [1].

3. In this section we prove our main results.

THEOREM 1. *For fixed $x \in [0,\infty)$ and all $T \geq 0$, we have*

$$(3.1)\qquad \sup_{0\leq t\leq T} |(P_{n,t}f)(x) - f(x+t)| \to 0 \text{ as } n \to \infty.$$

Proof. From the definition (1.2) we write

$$|(P_{n,t}f)(x) - f(x+t)| \leq \sum_{|\frac{k}{n^\alpha}-t|<\delta} A_{n,k,t}\left|f\left(x+\frac{k}{n^\alpha}\right) - f(x+t)\right| +$$
$$+ \sum_{|\frac{k}{n^\alpha}-t|\geq\delta} A_{n,k,t}\left|f\left(x+\frac{k}{n^\alpha}\right) - f(x+t)\right| = S_1 + S_2\ (say).$$

This first sum S_1 in the above expression is arbitrarily small if δ is chosen sufficiently small. The choice of δ depends only on powers of n.

Now, with $\delta > 0$ so chosen and fixed, and with $M = \sup_{x\geq 0}|f(x)|$, the following estimate

$$S_2 \leq \frac{2M}{\delta^2}\sum_{|\frac{k}{n^\alpha}-t|\geq\delta} A_{n,k,t}\left(\frac{k}{n^\alpha}-t\right)^2 \leq$$
$$\leq \frac{2M}{\delta^2}\sum_{k=0}^{n} A_{n,k,t}\left(\frac{k}{n^\alpha}-t\right)^2 = \frac{2M}{\delta^2}B_{n,\alpha,t}$$

(using the result (2.3)), approaches to 0 as $n \to \infty$.

Hence the theorem is proved.

THEOREM 2. *Let $f \in C^{(1)}[0,\lambda]$, $\lambda > 0$, be such that $w(f';\delta)$ is the modulus of continuity of f'. Then for $n \geq 1$ and $\delta > 0$ one gets*

$$(3.2)\ |(P_{n,t}f)(x) - f(x+t)| \leq \frac{n^{2\alpha-1}t^2}{1+n^{\alpha-1}t}\|f'\| + w(f';\delta)\{\sqrt{B_{n,\alpha,t}} + B_{n,\alpha,t}\}.$$

P r o o f. Using the mean value theorem, we write

$$f\left(x+\frac{k}{n^\alpha}\right)-f(x+t)=\left(\frac{k}{n^\alpha}-t\right)f'(x+t)+\left(\frac{k}{n^\alpha}-t\right)\{f'(x+\eta)-f'(x+t)\},$$

where η lies between t and $\frac{k}{n^\alpha}$.

Now, applying (1.2) on above, we get

$$|(P_{n,t}f)(x)-f(x+t)|=\left|\sum_{k=1}^{n}A_{n,k,t}\left[f\left(x+\frac{k}{n^\alpha}\right)-f(x+t)\right]\right|,$$

and, using the inequality

$$|f'(x+\eta)-f'(x+t)|\le\{1+|\eta-t|\delta^{-1}\}w(f';\delta), \tag{3.3}$$

we see that

$$|(P_{n,t}f)(x)-f(x+t)|\le|f'(x+t)|\left|\sum_{k=1}^{n}A_{n,k,t}\left(\frac{k}{n^\alpha}-t\right)\right|+ \tag{3.4}$$
$$+w(f';\delta)\left\{\sum_{k=1}^{n}A_{n,k,t}\left|\frac{k}{n^\alpha}-t\right|+\sum_{k=1}^{n}A_{n,k,t}\left(\frac{k}{n^\alpha}-t\right)^2\right\}.$$

Now, using the results (2.1) to (2.3) in the inequality (3.4), we get the required result (3.2).

Hence the theorem is proved.

Using a slight different method, we can get the following result.

THEOREM 3. *Under the conditions of Theorem 2, one gets*

$$|(P_{n,t}f)(x)-f(x+t)|\le \tag{3.5}$$
$$\le\left(\frac{n^{2\alpha-1}t^2}{1+n^{\alpha-1}t}\right)\|f'\|+w(f';\delta)\left\{\frac{x^2}{\delta}+\left(1+\frac{x}{\delta}\right)\sqrt{B_{n,\alpha,t}}+\left(\frac{B_{n,\alpha,t}}{2\delta}\right)\right\}.$$

P r o o f. We know that

$$f\left(x+\frac{k}{n^\alpha}\right)-f(x+t)=\left(\frac{k}{n^\alpha}-t\right)f'(t)+\int_{x+t}^{x+\frac{k}{n^\alpha}}(f'(y)-f'(t))\,dy.$$

Now, applying (1.2) and using the inequality (3.3) on above, we get

$$|(P_{n,t}f)(x)-f(x+t)|\le|f'(t)|\left|\sum_{k=1}^{n}A_{n,k,t}\left(\frac{k}{n^\alpha}-t\right)\right|+$$
$$+w(f';\delta)\sum_{k=1}^{n}A_{n,k,t}\dots\left|\int_{x+t}^{x+\frac{k}{n^\alpha}}\left\{1+\frac{|y-t|}{\delta}\right\}dy\right|\le$$

$$\leq |f'(t)|\left|\sum_{k=1}^{n} A_{n,k,t}\left(\frac{k}{n^{\alpha}}-t\right)\right| +$$

$$+ w(f';\delta)\sum_{k=1}^{n} A_{n,k,t}\left\{\frac{x^2}{\delta}+\left(1+\frac{x}{\delta}\right)\left|\frac{k}{n^{\alpha}}-t\right|+\ldots+\frac{1}{2\delta}\left(\frac{k}{n^{\alpha}}-t\right)^2\right\}.$$

Now using the results (2.1) to (2.3) in the above expression, we get the required result (3.5).

Hence the theorem is proved.

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